Section 5.2 Exponential and Logarithm Functions in Finance

Question 1 – How do you convert between the exponential and logarithmic forms of an equation?

Question 2 – How do you evaluate a logarithm?

Question 3 - How do you solve problems using logarithms?

Question 1 – How do you convert between the exponential and logarithmic forms of an equation?

Key Terms

Exponential form Logarithmic form

Summary

Exponential form and logarithmic form are different ways at looking inputs and outputs. The exponential function

$$y = 10^x$$

takes the variable *x* as its input and outputs the variable *y*. For an input of $x = 2$ we get an output of $y = 100$ since

$$100 = 10^2$$

On a logarithm of base 10 (called a common logarithm), these roles are reversed. The common logarithm must take in $y = 100$ and output $x = 2$,

$$2 = \log_{10}(100)$$

For common logarithms, those with base 10, the base on the logarithm is often left out and written as

$$2 = \log(100)$$

This means that whenever you see a logarithm without a base, it is assumed to have a base of 10. Let’s compare these forms side by side.

The same type of relationships exists for exponentials with a positive base and the corresponding logarithm with that base. For all bases $b > 0$,

$$y = b^x \text{ means that } x = \log_b(y)$$

<u>Notes</u>

Guided Example	Practice
Rewrite each exponential form in logarithmic form. a. $6^2 = 36$ **Solution** Start by recognizing the base on the exponential form, 6. This means the corresponding logarithmic form will be a logarithm base 6. Since the exponential from takes 2 as the input and outputs 36, the logarithmic form must take in 36 and output 2. This give the logarithmic form, $\log_6(36) = 2$ b. $3^{-4} = \frac{1}{81}$ **Solution** The base in the logarithmic form must be 3. Since the exponential form takes input -4 and outputs $\frac{1}{81}$, the logarithmic form must do the opposite, $\log_3\left(\frac{1}{81}\right) = -4$ c. $e^z = Y$ **Solution** The base on the exponential form is e so the corresponding logarithm is $\log_e$ or ln. The exponential form takes in z and out puts Y so the logarithmic form is $\ln(Y) = z$	1. Rewrite each exponential form in logarithmic form. a. $5^3 = 125$ b. $2^{-4} = \frac{1}{16}$ c. $e^{rt} = \frac{FV}{PV}$

Guided Example	Practice
Rewrite each logarithmic form in exponential form. a. $\log_5(125) = 3$ **Solution** The logarithm will convert to an exponential form with base 5. Since the logarithm takes in 125 and outputs 3, the exponential form is $$5^3 = 125$$ b. $\log(0.01) = -2$ **Solution** Since this is a common logarithm, the base is hidden (but equal to 10). Switching the input and outputs in exponential form leads to $$10^{-2} = 0.01$$ c. $\log\left(\frac{I}{I_0}\right) = R$ **Solution** In this common logarithm, the group of symbols $\frac{I}{I_0}$ form the input and R is the output. The corresponding exponential form is $$10^R = \frac{I}{I_0}$$	2. Rewrite each logarithmic form in exponential form. a. $\log_7(49) = 2$ b. $\log(0.001) = -3$ c. $\log\left(1+Z\right) = Q$

Question 2 – How do you evaluate a logarithm?

Key Terms

Summary

Many logarithms may be calculated by converting them to exponential form. Suppose we want to calculate the value of $\log_2(16)$. Start by writing this expression as a logarithmic form,

$$\log_2(16) = ?$$

We could write the output with a variable, but a question mark suffices to indicate what we want to find. If we convert this form to an exponential form with a base of 2,

$$2^{?} = 16$$

The left-hand side may be written with the base 2 as 2^4. Substitute this expression in place of 16,

$$2^{?} = 2^4$$

Since the exponent on the left side must be 4, this is also the value in the original exponential form,

$$\log_2(16) = 4$$

This strategy works well if we can write the number on the right with the same base as the exponential on the other side of the equation.

If you are not able to write the number on the right with the same base, we can evaluate common logarithms with the LOG key or a natural logarithm with the LN key on a calculator. If the base on the logarithm is not 10 (common logarithm) or e (natural logarithm), we use the Change of Base Formula to find the logarithm.

Change of Base Formula for Logarithms

For any positive base *a* and *b* not equal to 1,

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

where *x* is a positive number. Typically, the logarithms on the right-side are done as natural or common logarithms so they can be evaluated on any calculator.

<table>
<tr><th>Guided Example</th><th>Practice</th></tr>
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Evaluate each logarithm without a calculator.

a. $\log_4(64)$

Solution Write the logarithm in logarithmic form with ? forming the output of the logarithm,

$$\log_4(64) = ?$$
Convert this to exponential form,

$$4^? = 64$$
Since $4^3 = 64$, the value of ? is 3 and

$$\log_4(64) = 3$$
b. $\log_6\left(\frac{1}{36}\right)$

Solution Write the logarithm as

$$\log_6\left(\frac{1}{36}\right) = ?$$
In exponential form this becomes
$$6^? = \frac{1}{36}$$
We know that $6^2 = 36$ and negative powers lead to reciprocals so $6^{-2} = \frac{1}{36}$. The corresponding logarithmic form gives us the value of the logarithm,
$$\log_6\left(\frac{1}{36}\right) = -2$$
</td><td>
1. Evaluate each logarithm without a calculator.

a. $\log_8(64)$

b. $\log_3\left(\frac{1}{3}\right)$
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Guided Example	Practice
Evaluate $\log_3(32)$ using natural logarithms or common logarithms. **Solution** If we write $\log_3(32) = ?$ in logarithmic form, we get $$3^? = 32$$ Since 32 is not a power of 3, use the change of base formula with common logarithms to find the values, $$\log_3(32) = \frac{\log(32)}{\log(3)} \approx \frac{1.5051}{0.4771} \approx 3.155$$ Where the common logs are calculated on a calculator. Notice that we could have also done this with natural logarithms, $$\log_3(32) = \frac{\ln(32)}{\ln(3)} \approx \frac{3.4657}{1.0986} \approx 3.155$$	2. Evaluate $\log_6(37)$ using natural logarithms or common logarithms.

Question 3 – How do you solve problems using logarithms?

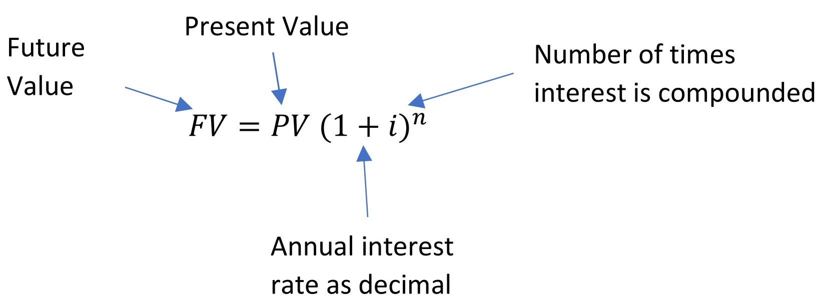
<u>Key Terms</u>

<u>Summary</u>

If a problem contains an exponential or logarithm function and you need to solve for something inside of the exponential function or logarithm function, converting forms may be useful to solve for the unknown.

Suppose \$5000 is deposited in an account that earns 2% compound interest that is done annually. In how many years will there be \$6000 in the account.

This problem requires the use of the compound interest formula,



Let's look at the quantities in the problem statement:

\$5000 is deposited in an account $\rightarrow$ $PV = 5000$

that earns 2% compound interest that is done annually $\rightarrow$ $i = 0.02$

Will there be \$6000 in the account $\rightarrow$ $FV = 6000$

Putting these values into the formula above gives us

$$6000 = 5000(1+0.02)^n$$

Divide both sides by 5000 to get the exponential piece by itself:

$$\frac{6000}{5000} = (1+0.02)^n$$

Now convert to logarithmic form:

$$\log_{1.02}\left(\frac{6000}{5000}\right) = n$$

Calculate the logarithm using common logarithms:

$$\frac{\log\left(\frac{6000}{5000}\right)}{\log(1.02)} = n$$

$$\frac{0.0792}{0.0086} \approx n$$

Using a calculator to do the logs, we get $n \approx 9.21$ years. Notice how this example requires you to convert to logarithm form and evaluate a logarithm with common logarithms.

<u>Notes</u>

Guided Example	Practice
Find the time required for \$5000 to grow to at least \$9100 when deposited at 2% compounded continuously. **Solution** Since interest is being compounded continuously, the future value is given by $$FV = PV\,e^{rt}$$ The future value is $FV = 9100$, the present value is $PV = 5000$, and the rate is $r = 0.02$. Put these values into the formula above to get $$9100 = 5000\,e^{0.02t}$$ Divide both sides by 5000 to put the equation in exponential form: $$\frac{9100}{5000} = e^{0.02t}$$ This converts to a logarithmic form, $$0.02t = \ln\left(\frac{9100}{5000}\right)$$ To solve for *t*, divide both sides by 0.02, $$t = \frac{\ln\left(\frac{9100}{5000}\right)}{0.02} \approx 29.94$$ In approximately 29.94 years, the \$5000 will have grown to \$9100.	1. Find the time required for \$2000 to double when deposited at 8% compounded continuously. This time is called the doubling time.

<u>Guided Example</u>

Monthly sales of a Blue Ray player are approximately

$$S(t) = 1000 - 750e^{-t} \text{ thousand units}$$

where t is the number of months the Blue Ray player has been on the market.

a. Find the initial sales.

Solution The initial sales occur at $t = 0$. The corresponding sales are

$$S(0) = 1000 - 750e^{-0} = 250 \text{ thousand units}$$

or 250,000 units.

b. In how many months will sales reach 500,000 units?

Solution Set $S(t)$ equal to 500 and solve for t.

$$500 = 1000 - 750e^{-t}$$

$$-500 = -750e^{-t}$$ Subtract 1000 from both sides

$$\frac{-500}{-750} = e^{-t}$$ Divide both sides by -750

$$\frac{2}{3} = e^{-t}$$ Reduce the fraction

$$-t = \ln\left(\tfrac{2}{3}\right)$$ Convert the exponential form to logarithm form

$$t = -\ln\left(\tfrac{2}{3}\right) \approx 0.41 \text{ months}$$ Multiply both sides by -1 and evaluate the logarithm

c. Will sales ever reach 1000 thousand units?

Solution Follow steps similar to part b.

$1000 = 1000 - 750e^{-t}$ Set $S(t)$ equal to 1000

$0 = -750e^{-t}$ Subtract 1000 from both sides

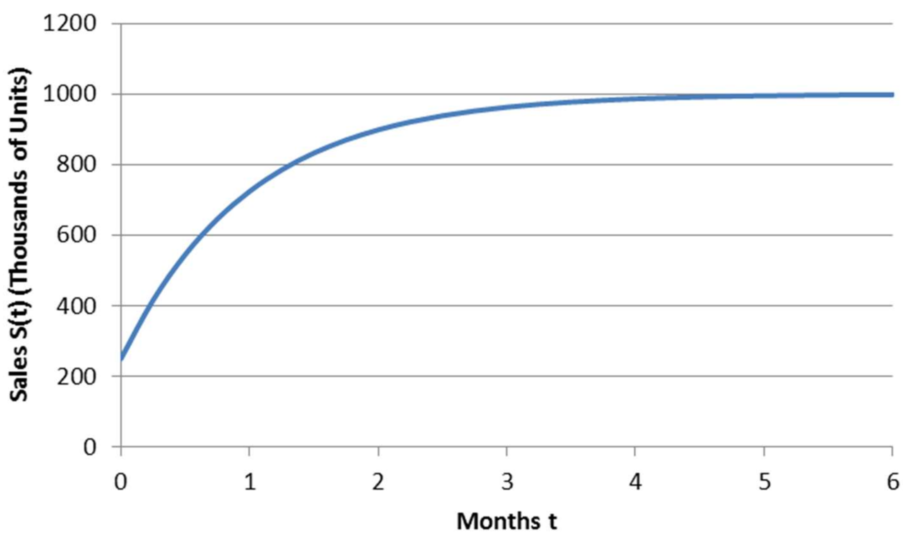
$0 = e^{-t}$ Divide both sides by -750

$-t = \ln(0)$ Convert the exponential form to logarithm form

Since the logarithm of zero is not defined, sales will never be 1000 thousand units.

d. Is there a limit for sales?

To help us answer this question, let's look at a graph of $S(t)$.



Examining the graph, it appears that the sales are getting closer and closer to 1000 units, but never quite get there (part c). So, the limit for sales is 1000 thousand units or 1,000,000 units.

<u>Practice</u>

2. Monthly sales of a package of mixed nuts are approximately

$$S(t) = 2000 - 1500e^{-t} \text{ thousand units}$$

where t is the number of years the mixed nuts has been on the market.

a. Find the initial sales.

b. In how many years will sales reach 1,000,000 units?

c. Will sales ever reach 2100 thousand units?

d. Is there a limit for sales?