Section 7.2 Probability Rules

Question 1 – How do you find the probability of an event?

Question 2 – What is the complement of an event?

Question 3 - How do you find the probability of a compound event?

Question 4 – What is the difference between marginal and joint probability?

Question 1 – How do you find the probability of an event?

Key Terms

Event

Summary

An **event** is any collection of outcomes in the sample space. The probability of an event is the sum of the probabilities of the outcomes corresponding to the event.

Probability of an Event

If *E* is composed of a collection of the outcomes in the sample space *S*,

$$E=\{e_1,e_2,...,e_n\}$$

then the probability of the event *E* is

$$P(E)=P(e_1)+P(e_2)+\cdots+P(e_n)$$

We can use the probabilities of the outcomes in the sample space to find the probability of any event.

The probability of events composed of equally likely outcomes may be calculated by counting the number of outcomes in the event. For instance, if an event contains *M* outcomes (each with probability $\frac{1}{N}$), the probability of the event must be

$$P(E)=\underbrace{\frac{1}{N}+\frac{1}{N}+\cdots+\frac{1}{N}}_{M\text{ terms}}=\frac{M}{N}$$

Probability of an Event With Equally Likely Outcomes

Suppose the sample space of an experiment contains *N* equally outcomes. If an event *E* contains *M* of those outcomes, the probability of the event is

$$P(E) = \frac{M}{N}$$

This expression must be used cautiously since it requires that each outcome in the sample space be equally likely. This expression is often written using the letter *n* to indicate the number of outcomes in a collection. In this case,

$$P(E) = \frac{n(E)}{n(S)}$$

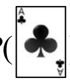
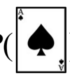
where the notations $n(E)$ and $n(S)$ are the number of outcomes in the event *E* and sample space *S*.

<u>Notes</u>

Guided Example 1

You are dealt one card from a standard 52-card deck. Find the probability of being dealt an ace.

Solution The likelihood of dealing any individual card in the deck is $\frac{1}{52}$. Since there are four aces in the deck, the probability of dealing an ace is

$P(\text{ace}) = P($$) + P($$) + P($$) + P($$)$

Each of these cards has a probability of $\frac{1}{52}$ and we can insert these probabilities to give

$$P(\text{ace}) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52}$$

We could also calculate this probability by counting the aces,

$$P(\text{ace}) = \frac{\text{number of aces in deck}}{\text{number of cards in deck}} = \frac{4}{52}$$

This probability simplifies to $\frac{1}{13}$ or approximately 7.7%.

Practice

You are dealt one card from a standard 52-card deck. Find the probability of being dealt a heart.

Guided Example 2

A marble is selected at random from a jar containing 3 red marbles, 2 yellow marbles, and 5 green marbles.

a. What is the probability that the marble is red?

Solution Each of ten marbles is equally likely. The probability is calculated by counting the outcomes,

$$P(\text{red}) = \frac{n(\text{red})}{n(S)} = \frac{3}{10}$$

The probability of selecting a red marble is 0.3 or 30%.

Practice

A marble is selected at random from a jar containing 5 red marbles, 13 purple marbles, and 2 green marbles.

a. What is the probability that the marble is purple?

b. What is the probability that the marble is green? **Solution** There are five green marbles, $$P(\text{green}) = \frac{n(\text{green})}{n(S)} = \frac{5}{10}$$ The probability of selecting a green marble is 0.5 or 50%.	b. What is the probability that the marble is green?

Guided Example 3

The table below lists the estimated number of preventable injuries in the United States in 2016 and their causes.

Cause	Injuries
Fall	8,591,683
Struck By/Against	3,685,012
Overexertion	2,569,850
Motor vehicle-Occupant	2,500,353
Other Specified	2,365,891
Cut/Pierce	1,823,358
Poisoning	1,755,044
Other Bite/Sting	1,142,130
Unknown/Unspecified	755,567
Foreign Body	557,650

Source: Center for Disease Control and Prevention, National Center for Injury Prevention and Control

a. What is the probability that a randomly selected person with a preventable injury was hurt by overexertion?

Solution To find the probability of preventable injury by overexertion, we must first find the total number of preventable injuries in the table. Adding the number of preventable injuries gives a total of 25,746,538 injuries. The probability that a preventable injury was caused by overexertion is the number of preventable overexertion injuries

Practice

The table below lists the estimated number of preventable deaths in the United States in 2016 and their causes.

Cause	Deaths
Poisoning	64,795
Motor vehicle	38,659
Fall	36,338
Suffocation	6,946
Drowning	3,709
Fire/burn	2,902
Natural/ Environment	1,750
Other Specified, classifiable	1,440
Other Land Transport	1,332
Other Specified	1,251

Source: Center for Disease Control and Prevention, National Center for Injury Prevention and Control

a. What is the probability that a randomly selected person with a preventable injury died by motor vehicle?

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divided by the total number of preventable injuries,

$$P(\text{overexertion}) = \frac{2569850}{25746538} \approx 0.100$$

Written as a percent, this means the likelihood that a preventable injury was caused by overexertion is 10.0%.

b. What is the probability that a randomly selected person with a preventable injury was hurt by “other”?

Solution Two of the injuries listed in the table include the term “other” (Other Specified and Other Bite/Sting). The number of “other” causes is 2,365,891 + 1,142,130 or 3,508,021. The probability is calculated by dividing this number by the total number of preventable injuries,

$$P(\text{other}) = \frac{3508021}{25746538} \approx 0.136$$

This means that the likelihood of a preventable injury being caused by “other” is 13.6%.
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b. What is the probability that a randomly selected person with a preventable injury died “other”?
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Question 2 – What is the complement of an event?

Key Terms

Complement

Summary

The complement of an event E is all of the outcomes in the sample space that are not in the event E. The complement of an event E is represented by the symbols E'. In discussing the complement, we often refer to it as the outcomes not in E. Since the event E and the event not in E combine to give the entire sample space S,

$$P(E)+P(E')=P(S)$$

The likelihood of an outcomes in the sample space occurring is certain, so we simplify this to

$$P(E)+P(E')=1$$

This leads us to a convenient relationship for determining the likelihood that an outcome in the complement will occur.

Probability of the Complement of an Event

The probability that an outcome in the complement E' will occur is

$$P(E')=1-P(E)$$

In other words, the probability that an event will not occur is 1 minus the probability that it will occur.

Notes

Guided Example 4	Practice
You are dealt one card from a standard 52-card deck. Find the probability of being dealt a card that is <u>not</u> an ace. **Solution** From Guided Example 1 in this section, we know that $$P(\text{ace}) = \frac{\text{number of aces in deck}}{\text{number of cards in deck}} = \frac{4}{52}$$ To find the probability of not being dealt an ace, we realize that being dealt an ace and not being dealt an ace are compliments of each other. We can subtract the probability of being dealt an ace from 1 to get the probability of not being dealt an ace, $$P(\text{not ace}) = 1 - P(\text{ace})$$ $$= 1 - \frac{4}{52}$$ $$= \frac{48}{52}$$ or approximately 92.3%.	You are dealt one card from a standard 52-card deck. Find the probability of being dealt a card that is <u>not</u> a heart.

Guided Example 5	Practice
A marble is selected at random from a jar containing 3 red marbles, 2 yellow marbles, and 5 green marbles. a. What is the probability that the marble is <u>not</u> red? **Solution** From Guided Example 2 in this section, $$P(\text{red}) = \frac{n(\text{red})}{n(S)} = \frac{3}{10}$$ Since the event "marble selected is red" and "marble selected is not red" are compliments of each other, $$\begin{aligned} P(\text{not red}) &= 1 - P(\text{red}) \\ &= 1 - \frac{3}{10} \\ &= \frac{7}{10} \end{aligned}$$ b. What is the probability that the marble is <u>not</u> green? **Solution** In Guided Example 2, we calculated $$P(\text{green}) = \frac{n(\text{green})}{n(S)} = \frac{5}{10}$$ Since the event "marble selected is green" and "marble selected is not green" are compliments, $$\begin{aligned} P(\text{not green}) &= 1 - P(\text{green}) \\ &= 1 - \frac{5}{10} \\ &= \frac{5}{10} \end{aligned}$$	A marble is selected at random from a jar containing 5 red marbles, 13 purple marbles, and 2 green marbles. a. What is the probability that the marble is <u>not</u> purple? b. What is the probability that the marble is <u>not</u> green?

Guided Example 6

The table below lists the estimated number of preventable injuries in the United States in 2016 and their causes.

Cause	Injuries
Fall	8,591,683
Struck By/Against	3,685,012
Overexertion	2,569,850
Motor vehicle-Occupant	2,500,353
Other Specified	2,365,891
Cut/Pierce	1,823,358
Poisoning	1,755,044
Other Bite/Sting	1,142,130
Unknown/Unspecified	755,567
Foreign Body	557,650

Source: Center for Disease Control and Prevention, National Center for Injury Prevention and Control

What is the probability that a randomly selected person with a preventable injury was hurt by a cause other than overexertion?

Solution The event "selected person with a preventable injury was hurt by a cause other than overexertion" corresponds to all of the rows in the table except the overexertion table. This helps us to deduce that the event "selected person with a preventable injury was hurt by a cause other than overexertion" and the event "selected person with a preventable injury was hurt by overexertion" are compliments of each other. We can use the results from Guided Example 3 to calculate

$$\begin{aligned} P(\text{not overexertion}) &= 1 - P(\text{overexertion}) \\ &= 1 - \frac{2569850}{25746538} \\ &= \frac{23176688}{25746538} \\ &\approx 0.900 \end{aligned}$$

or approximately 90.0%.

Practice

The table below lists the estimated number of preventable deaths in the United States in 2016 and their causes.

Cause	Deaths
Poisoning	64,795
Motor vehicle	38,659
Fall	36,338
Suffocation	6,946
Drowning	3,709
Fire/burn	2,902
Natural/ Environment	1,750
Other Specified, classifiable	1,440
Other Land Transport	1,332
Other Specified	1,251

Source: Center for Disease Control and Prevention, National Center for Injury Prevention and Control

What is the probability that a randomly selected person with a preventable death by a cause other than motor vehicle?

Question 3 – How do you find the probability of a compound event?

Key Terms

Compound event Intersection Union

Union rule for probability

Summary

Events may be combined together in various ways. These combinations are called **compound events**. If we know the probability of the events that make up the compound event, we are often able to compute the probability of the compound event.

Outcomes that are in the event *A* as well as the event *B* are said to be in the compound event, *A* and *B*. The word "and" is used to indicate that the outcomes in this event are in both events simultaneously. Mathematicians describe outcomes in *A* and *B* with the intersection symbol $\cap$. An outcome in *A* and *B* are the same outcomes in the intersection of *A* with *B*, $A \cap B$. The probability of *A* and *B* occurring is often referred to as the joint probability of *A* and *B*.

Another type of compound event is denoted using the word "or". An outcome is in the event *A* or *B* if it is in *A*, *B*, or both events simultaneously. In the language of sets, the compound event *A* or *B* is the same as the union of the set *A* with the set *B*. The symbol $\cup$ represents the union of two sets. Using this symbol, we write the union of *A* with *B* as $A \cup B$. In this text we will use the word "or" instead of the union symbol to represent outcomes in *A*, in *B*, or in both events.

Compound probabilities are related to each other the following relationship.

The Probability of A or B

The likelihood of an event A occurring or an event B occurring is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This rule is often referred to as the **union rule for probability**.

Notes

Guided Example 7

A product survey of form returned by 1000 consumers after the purchase of a new vacuum cleaner gives the following results.

Event	Number of consumers
Consumer feels the vacuum is a good value	950
Consumer feels vacuum's manual is easy to follow	430
Consumer feels the vacuum is a good value and vacuum's manual is easy to follow	400

Define the events

G: Consumer feels the vacuum is a good value
M: Consumer feels the vacuum's manual is easy to follow.

a. Write the compound event "Consumer feels the vacuum is a good value and vacuum's manual is easy to follow" in terms of the events defined above.

Solution The compound event uses the word "and" to connect the events. This corresponds to the joint event G and M.

b. Describe the event G or M in words.

Solution The event G or M corresponds to outcomes in which the consumer feels the vacuum is a good value or feels the manual is easy to follow.

Practice

A mechanic tracks 100 repairs with the following results.

Event	Number of repairs
Repair requires a new battery	50
Repair requires a new alternator	10
Repair requires a new battery and a new alternator	5

Define the events

B: Repair requires a new battery
A: Repair requires a new alternator

a. Write the compound event "needs a new battery and a new alternator" in terms of the events defined above.

b. Describe the event B or A in words.

c. What is the probability of G or M occurring? **Solution** To find $P(G \text{ or } M)$, apply $P(G \text{ or } M) = P(G) + P(M) - P(G \text{ and } M)$ Each of the probabilities on the right may be calculated from relative frequencies: $P(G) = \frac{950}{1000} = 0.95$ $P(M) = \frac{430}{1000} = 0.43$ $P(G \text{ and } M) = \frac{400}{1000} = 0.4$ Using these probabilities gives $P(G \text{ or } M) = 0.95 + 0.43 - 0.40 = 0.98$ or 98%	c. What is the probability of B or A occurring?

Guided Example 8

What is the probability of getting either a sum of 6 or doubles in the roll of a pair of dice?

Solution Let's define the events as follows:

A: sum of six
B: doubles

The event "getting either a sum of 6 or doubles" corresponds to A or B. The union rule for probability states that

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

A table of all possible rolls of two dice helps us to calculate each of the probabilities on the right side.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

In this table, the faces of each die are indicated in the shaded portion and the corresponding sums in the unshaded portion.

To find $P(A)$, we locate the combinations of the dice that sum to 6.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 36 possible rolls of the dice and five of them sum to 6 so

Practice

What is the probability of getting either a sum of 8 or doubles in the roll of a pair of dice?

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

The likelihood of doubles is found by counting the possibilities in the table.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 6 ways to get doubles,

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36}$$

The event A and B corresponds to all rolls of the dice with a sum of 6 and that are doubles. This event corresponds to one out come where each die is a three:

$$P(A \text{ and } B) = \frac{n(A \text{ and } B)}{n(S)} = \frac{1}{36}$$

Put these probabilities into the union rule to give

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= \frac{5}{36} + \frac{6}{36} - \frac{1}{36} \\ &= \frac{10}{36} \end{aligned}$$

or approximately 27.8%.

Question 4 – What is the difference between marginal and joint probability?

Key Terms

Marginal probability

Summary

In Question 3 we introduced the idea of joint probability. Joint probabilities are the likelihoods associated with compound events using "and". The joint probability of A and B is the likelihood that both events will occur simultaneously. **Marginal probabilities** are the probabilities of the individual events that make up the joint probability. Marginal probabilities are often found by using relative frequencies or the relationship

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

where A and B are events.

Notes

Guided Example 9

Suppose a cellular provider collects the data below about a sample of users.

Amount of Data Used	Male Users	Female Users
Less than 1GB	425	300
1 GB up to, but not including 2GB	1505	1225
2 GB up to, but not including 5 GB	450	550
5 GB up to, but not including 10GB	330	625
10 GB or more	125	205

a. Find the likelihood that a female user will use more than 10 GB of data.

Solution To make the problem easier to complete, let's sum the row and columns of the table.

Amount of Data Used	Male Users	Female Users	Total
Less than 1GB	425	300	725
1 GB up to, but not including 2GB	1505	1225	2730
2 GB up to, but not including 5 GB	450	550	1000
5 GB up to, but not including 10GB	330	625	955
10 GB or more	125	205	330
Total	2835	2905	5740

Define the events

F: user is female

A: users uses more than 10 GB of data

Practice

Suppose a cellular provider collects the data below about a sample of users.

Amount of Data Used	Male Users	Female Users
Less than 1GB	425	300
1 GB up to, but not including 2GB	1505	1225
2 GB up to, but not including 5 GB	450	550
5 GB up to, but not including 10GB	330	625
10 GB or more	125	205

a. Find the likelihood that a male user will use less than 1 GB of data.

In terms of these events, we must find the joint probability that the user is female and uses more than 10 GB, $P(F \text{ and } A)$. From the table, we recognize that there are 205 female users who use more than 10 GB of data. Since the total number of users is 5740, the relative frequency may be calculated, $$P(F \text{ and } A) = \frac{205}{5740} \approx 0.036$$ The likelihood that a user in the survey is female and used more than 10 GB of data is approximately 3.6%.	
b. Find the probability that a user is female. **Solution** To use calculate the relative frequency of the event, we must divide the number of female users by the total number of users. The total number female users is at the bottom of the third column. Dividing this by the total number of users in the bottom of the last column gives, $$P(F) = \frac{2905}{5740} \approx 0.506$$ The marginal probability that a user in the survey is female is approximately 50.6%.	b. Find the probability that a user is male.
c. Find the probability that a user in the survey will use more than 10 GB of data. **Solution** According to the survey, 330 users of the total 5740 users used more than 10 GB of data. This means the probability of using more than 10 GB of data is $$P(A) = \frac{330}{5740} \approx 0.057$$ The likelihood of using more than 10 GB of data is approximately 5.7%.	c. Find the probability that a user in the survey will use less than 1 GB of data.

d. Find the probability that the user is female or more than 10 GB of data is used. **Solution** The event "the user is female or more than 10 GB of data is used" corresponds to the compound event F or A. We calculate the probability of this event by using the probabilities of the events in parts a through c, $P(F \text{ or } A) = P(F) + P(A) - P(F \text{ and } A)$ $= \frac{2905}{5740} + \frac{330}{5740} - \frac{205}{5740}$ $= \frac{3030}{5740}$ ≈ 0.528 The probability that a user is female or uses more than 10 GB of data is approximately 52.8%.	d. Find the probability that the user is male or less than 1 GB of data is used.