Section 7.3 Conditional Probability

Question 1 – What is conditional probability?

Question 2 – How is conditional probability computed?

Question 3 - What are independent events?

Question 4 – What is the product rule for probability?

Question 5 - How is Bayes' Rule used to compute conditional probability?

Question 1 – What is conditional probability?

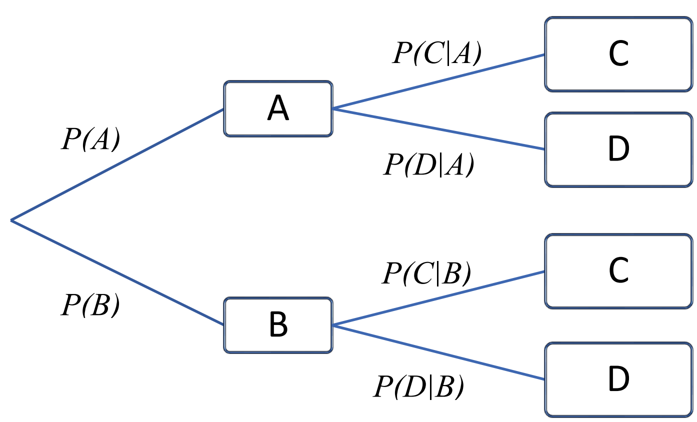
Key Terms

Conditional probability

Summary

Conditional probability is the likelihood of an event occurring given that another event has occurred. A vertical line is used to indicate the event whose probability is being computed and the event that has already occurred. For instance, the symbols $P(A \mid B)$ correspond to the probability of A occurring given that B has already occurred. The vertical bar separates the probability we are interested in calculating from the event that is assumed to have occurred.

A tree diagram is often used to represent conditional probabilities.



In this context, the events are depicted in the boxes and the corresponding probabilities are labeled on the branches connecting the boxes. If you follow the set of branches to A and then C, note that the first branch is labeled with $P(A)$ indicating the probability of A. Continuing to C, we see that the branch is labeled $P(C \mid A)$ indicating the probability of C given that A has occurred.

Guided Example 1

A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

Explain what each of the probabilities below mean.

a. $P(SP)$

Solution Since SP is the event "consumer owns a smartphone", $P(SP)$ is the probability that a consumer owns a smartphone. To find this probability, we simply count the number of consumers in this event and divide by the total number of consumers who took the survey:

$$P(SP) = \frac{n(SP)}{n(S)} = \frac{2802}{3300}$$

b. $P(F)$

Solution The event F corresponds to the event "consumer is female". The probability that a consumer in the survey is female is $P(F)$. This probability is found by dividing the number of males in the survey by the total number of consumers in the survey,

$$P(F) = \frac{n(F)}{n(S)} = \frac{1852}{3300}$$

Practice

A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

Explain what each of the probabilities below mean.

a. $P(M)$

b. $P(BP)$

c. $P(SP \text{ and } F)$ **Solution** The event *SP* and *F* is all of the outcomes in common between "consumer owns a smartphone" and consumer is female". So, $P(SP \text{ and } F)$ is the probability of a female consumer who owns a smartphone. To find this probability, divide the number of female consumers who own a smartphone by the total number of consumers in the survey, $P(SP \text{ and } F) = \frac{n(SP \text{ and } F)}{n(S)} = \frac{1601}{3300}$	c. $P(M \text{ and } BP)$
d. $P(SP \mid F)$ **Solution** The vertical bar tells us that we are describing an event with conditional probability. In this case, we are given the event that the "consumer is female" and we want to know the likelihood that the "consumer owns a smartphone". To find the probability of this event, we need to recognize that we are not interested in all consumers in the survey, only the female consumers. A total of 1852 female consumers took the survey. Of those female consumers, 1601 owned a smartphone. The probability of a consumer owning a smartphone given they are female is $P(SP \mid F) = \frac{1601}{1852}$	d. $P(M \mid BP)$

<u>Guided Example 2</u>

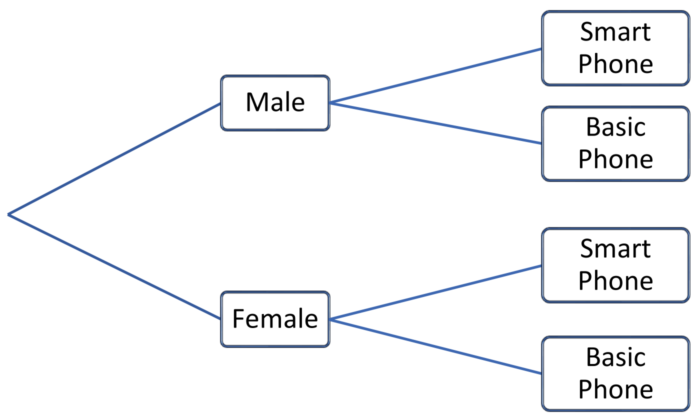
A survey was administered to a group of consumers who own a mobile phone. The results of the survey are below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

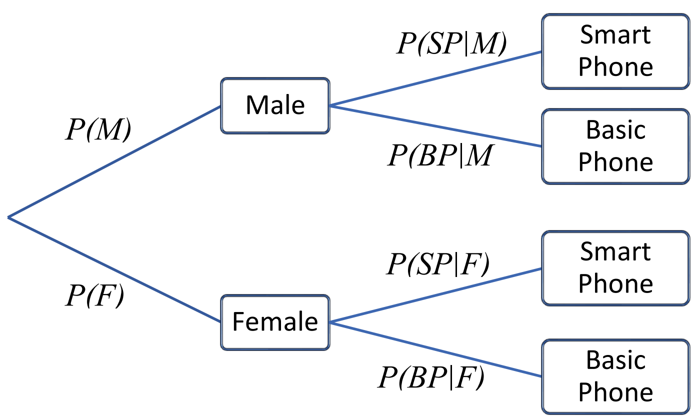
Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

Find the probabilities on each branch of the tree diagram below.



Solution In the tree diagram, we start on the left and work to the right. The branches are labeled with the probabilities below:



Examine the diagram carefully to note that the second level of the tree consists of conditional probabilities. In each case, the given part is where the branch originates and the probability we want is where the branch terminates.

To find the first set of probabilities, calculate the relative frequencies of males and females in the survey:

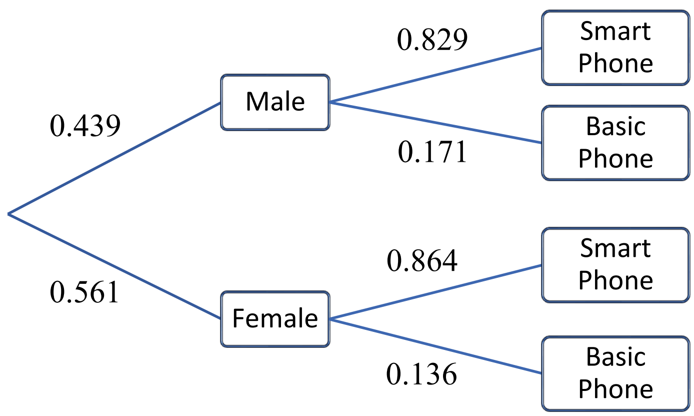
$$P(M) = \frac{1448}{3300} \approx 0.439, \quad P(F) = \frac{1852}{3300} \approx 0.561$$

The conditional probabilities are found by taking into account the given condition that the consumer is male or the consumer is female:

$$P(SP \mid M) = \frac{1201}{1448} \approx 0.829, \quad P(BP \mid M) = \frac{247}{1448} \approx 0.171$$

$$P(SP \mid F) = \frac{1601}{1852} \approx 0.864, \quad P(BP \mid F) = \frac{251}{1852} \approx 0.136$$

Label these probabilities on the tree diagram to give



Practice

A survey was administered to a group of consumers who own a mobile phone. The results of the survey are below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
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Define the events below:

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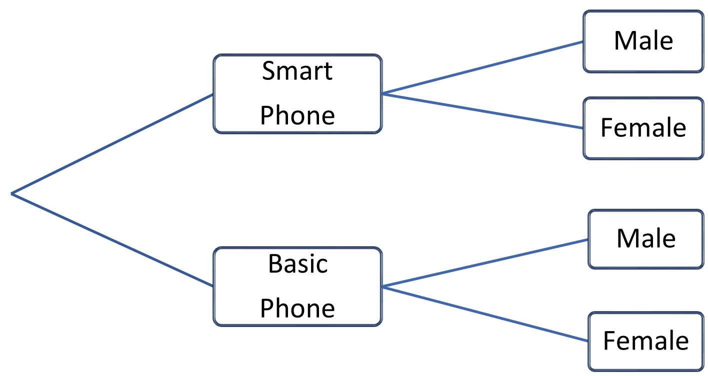
Find the probabilities on each branch of the tree diagram below.



Question 2 – How is conditional probability computed?

Key Terms

Summary

In Guided Example 1 of this section, we computed a conditional probability from the data below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

We defined the events follows:

M: Consumer is male

F: Consumer is female

BP: Consumer owns a basic phone

SP: Consumer owns a smart phone

To find the conditional probability $P(SP \mid F)$, we recognized that we are not interested in all consumers in the survey, only the female consumers. A total of 1852 female consumers took the survey. Of those female consumers, 1601 owned a smartphone. The probability of a consumer owning a smartphone given they are female is

$$P(SP \mid F) = \frac{1601}{1852}$$

Let's look at these numbers more closely. The denominator is the number of female consumers in the survey, $n(F)$. The numerator corresponds to female consumers who own a smartphone, $n(SP \text{ and } F)$. In words, these are consumers who are female and own a smartphone.

$n(SP \text{ and } F)$

$$P(SP \mid F) = \frac{1601}{1852}$$

$n(F)$

If we divide the top and bottom of this fraction by the number of people who took the survey,

$$P(SP \mid F) = \frac{\frac{1601}{3300}}{\frac{1852}{3300}}$$

we see that the top and the bottom are now relative frequencies and can be written as

$$P(SP \mid F) = \frac{P(SP \text{ and } F)}{P(F)}$$

This relationship allows us to write a conditional probability in terms of joint and marginal probabilities. In general, we can compute conditional probability with this relationship.

Conditional Probability

If *A* and *B* are events, the likelihood of *A* occurring given that *B* has occurred is

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

provided that $P(B) \neq 0$.

<u>Notes</u>

Guided Example 3	Practice
Suppose a batch of batteries is produced at a factory. A sample of batteries coming off the production line are sampled. From this sample, 32% of the batteries are mislabeled and 42% provide inadequate current. Twenty percent of the batteries are mislabeled and provide inadequate current. a. What is the probability that a battery provides inadequate current given that the battery is mislabeled? **Solution** Start by defining the events in the problem: M: Battery is mislabeled C: Battery provides inadequate current We can match the probabilities given in problem with events: $P(M) = 0.32$ $P(C) = 0.42$ $P(C \text{ and } M) = 0.20$ The question asks us to find the conditional probability $P(C \mid M)$. This may be found using the formula $$P(C \mid M) = \frac{P(C \text{ and } M)}{P(M)}$$ Substitute the probabilities given in the problem to get $$P(C \mid M) = \frac{0.20}{0.32} = 0.625$$ b. What is the probability the battery is mislabeled given that the battery provides inadequate current? **Solution** To find the probability $P(M \mid C)$, apply the formula for conditional probability and substitute the values given in the problem:	A mathematically inclined auto mechanic determines that 35% of repairs involve dead batteries and 15% of repairs involve bad alternators. Five percent of repairs involve dead batteries and bad alternators. a. What is the probability that a repair involves a dead battery given that the repair involves a bad alternator? b. What is the probability the repair involves a bad alternator given that the repair involves a dead battery?

$P(M \mid C) = \frac{P(M \text{ and C})}{P(C)}$ $= \frac{0.20}{0.42}$ ≈ 0.476	

Question 3 – What are independent events?

Key Terms

Independent events

Summary

When two events are **independent events**, one event occurring has no effect on the likelihood of the other event occurring.

Independent Events

If one event occurring does not change the likelihood of another event occurring, the two events are independent. This means that for events *A* and *B*,

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B)$$

In each case, the given events do not change the likelihood of the other events occurring. If one or both of the relationships are not equal, the events are said to be dependent.

Notes

Guided Example 4

A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

Are the events SP and F independent events?

Solution In order for SP and F to be independent events,

$$P(SP \mid F) = P(SP) \text{ and } P(F \mid SP) = P(F)$$

We can calculate each of these four probabilities using the numbers in the table above.

$$P(SP) = \frac{n(SP)}{n(S)} = \frac{2802}{3300} \approx 0.849$$

$$P(SP \mid F) = \frac{n(SP \text{ and } F)}{n(F)} = \frac{1601}{1852} \approx 0.864$$

and

$$P(F) = \frac{n(F)}{n(S)} = \frac{1852}{3300} \approx 0.561$$

$$P(F \mid SP) = \frac{n(F \text{ and SP})}{n(SP)} = \frac{1601}{2802} \approx 0.571$$

In both cases, the probabilities are not equal, so the events are dependent. For the events to be independent, both equations would need to be true.

Practice

A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

Are the events BP and M independent events?

Question 4 – What is the product rule for probability?

Key Terms

Product Rule for Probability

Summary

The rule for computing conditional property can be interpreted different. In Question 2, we defined the conditional probability $P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$. If we multiply both side of this equation by $P(B)$, we get

$$P(A \mid B)P(B) = P(A \text{ and } B)$$

We can also apply this strategy to the conditional probability $P(B \mid A) = \dfrac{P(B \text{ and } A)}{P(A)}$ to obtain a similar expression,

$$P(B \mid A)P(A) = P(B \text{ and } A)$$

These expressions give the joint probability of *A* and *B* as a product of a conditional probability and a marginal probability.

Product Rule for Probability

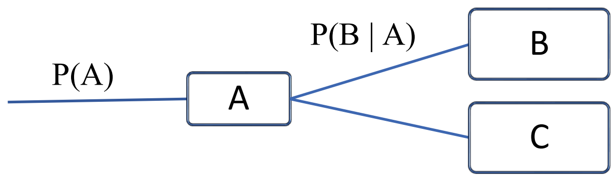
The probability of the event *A* and *B* is

$$P(A \text{ and } B) = P(A \mid B)P(B)$$

or

$$P(A \text{ and } B) = P(B \mid A)P(A)$$

We can utilize these relationships when we use a tree diagram. The probabilities on the right side of the second rule, $P(A \text{ and } B) = P(B \mid A)P(A)$, lie along the branch connecting to A followed by B. This means we can find the probability of A and B by multiplying the probabilities that connect to A followed by B.



Product Rule for Tree Diagrams

The product of all probabilities along a branch on a tree diagram is the likelihood of all events occurring that are on the branch.

Notes

Guided Example 5

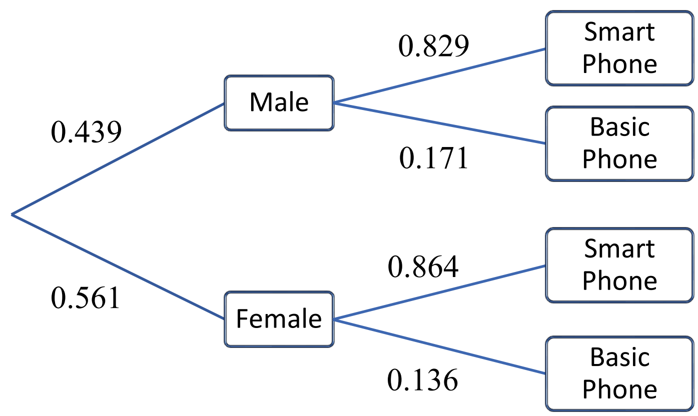
A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
F: Consumer is female
BP: Consumer owns a basic phone
SP: Consumer owns a smart phone

In an earlier example, we used these data and events to create the tree diagram below.



Compute the likelihood that a male consumer owns a smart phone.

Solution In terms of the events, we are being asked to find $P(M \text{ and } SP)$. Apply the formula for finding intersections of events to give

$$P(M \text{ and } SP) = P(SP \mid M)P(M)$$

The probabilities on the right side are found along the branch through Male and Smart Phone. Put these into the formula to yield

$$P(M \text{ and } SP) = (0.829)(0.439) \approx 0.364$$

Practice

A survey is administered to a group of consumers who own a mobile phone. The results of the survey are shown below.

	Male	Female	Total
Basic Phone	247	251	498
Smart Phone	1201	1601	2802
Total	1448	1852	3300

Define the events below:

M: Consumer is male
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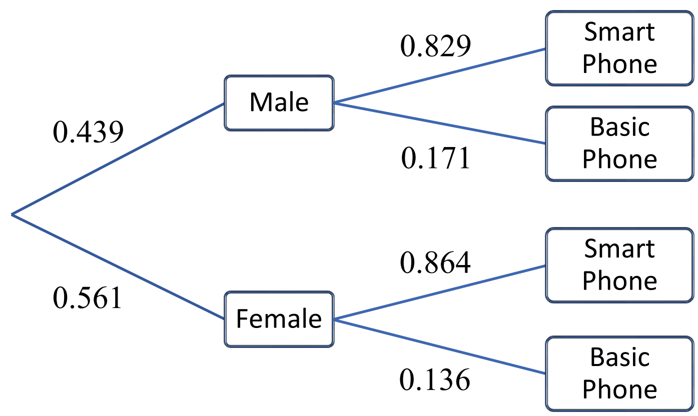
In an earlier example, we used these data and events to create the tree diagram below.



Compute the likelihood that a female consumer owns a basic phone.

Question 5 – How is Bayes' Rule used to compute conditional probability?

Key Terms

Bayes' Rule

Summary

In Question 2, we learned that the likelihood of an event A occurring given that an event B has already occurred is

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

We can also use the same basic expression to find the likelihood of an event B occurring given that an event A has already occurred,

$$P(B \mid A) = \frac{P(B \text{ and } A)}{P(A)}$$

Each of these expressions may be solved for the joint probability in the numerator to give

$$P(A \mid B)P(B) = P(A \text{ and } B)$$

$$P(B \mid A)P(A) = P(B \text{ and } A)$$

The joint event A and B is exactly the same event as the joint event B and A. This means their probabilities are also the same. Setting the left sides of these expressions equal gives

$$P(A \mid B)P(B) = P(B \mid A)P(A)$$

We can solve for either conditional probability, but if we solve for $P(B \mid A)$ we get the most basic form of **Bayes' Rule**.

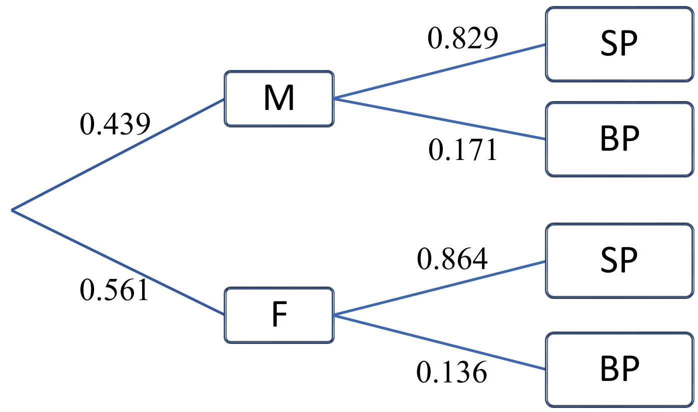
> **Bayes' Rule**
>
> If A and B are events, the conditional probability $P(B \mid A)$ may be computed in terms of $P(A \mid B)$ using
>
> $$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

This expression allows us to compute one conditional probability in terms of the "reverse" conditional probability. In practice, the most challenging part of using Bayes' Rule is identifying the events and computing the probabilities on the right side. We can simplify this task using a tree diagram.

Notes

<u>Guided Example 6</u>

Suppose you have the tree diagram below.



Use the tree diagram to compute $P(M \mid SP)$.

Solution Start by writing out the joint probabilities from the product rule for probabilities:

$$P(M \text{ and } SP) = P(M \mid SP)P(SP)$$
$$P(SP \text{ and } M) = P(SP \mid M)P(M)$$

Since the joint probabilities on the left side are equal, the expression on the left must be equal,

$$P(M \mid SP)P(SP) = P(SP \mid M)P(M)$$

Solving for $P(M \mid SP)$ gives the appropriate Bayes' Rule,

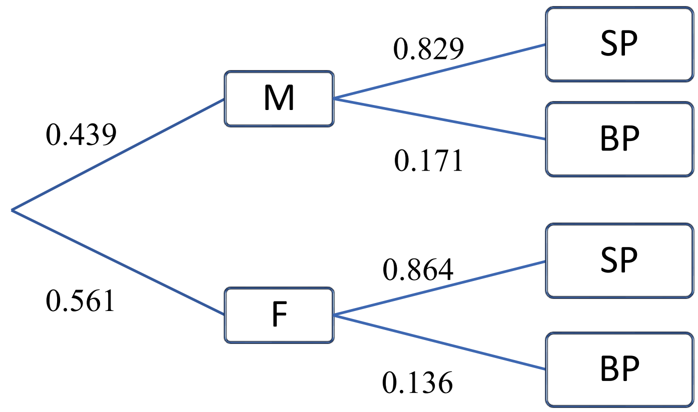
$$P(M \mid SP) = \frac{P(SP \mid M)P(M)}{P(SP)}$$

The probabilities for the numerator lie along the branch to M that continues to SP. To find the marginal probability in the denominator, locate all of the branches that terminate at SP. Multiply along these branches and then add the product to find the probability of SP. Putting this into Bayes' Rule yields

$$P(M \mid SP) = \frac{(0.829)(0.439)}{(0.829)(0.439)+(0.864)(0.561)} \approx 0.428$$

Practice

Suppose you have the tree diagram below.



Use the tree diagram to compute $P(F \mid BP)$.

Guided Example 7

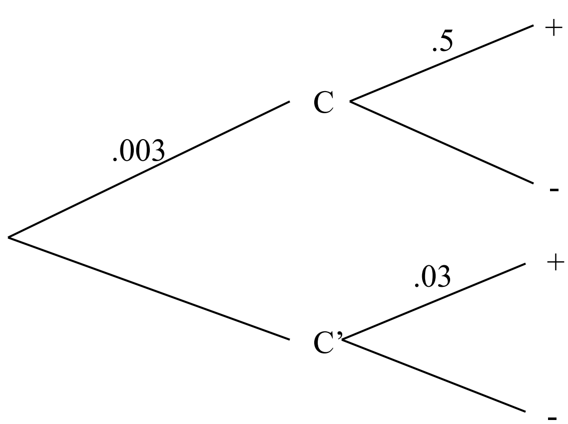
The probability of colorectal cancer can be given as .3%. If a person has colorectal cancer, the probability that the hemoccult test is positive is 50%. If a person does not have colorectal cancer, the probability that he still tests positive is 3%. What is the probability that a person who tests negative does not have colorectal cancer?

Solution To solve this problem, we'll draw and label an appropriate tree diagram. Then we'll apply Bayes' Rule to the problem.

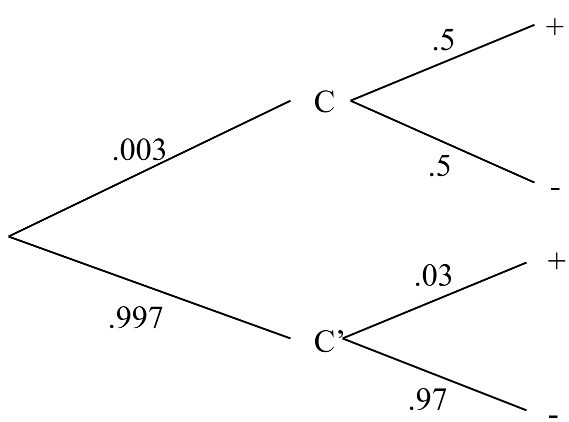
Look at the information given in the problem. If C is the event "person has colorectal cancer" and + is the event "the hemoccult test is positive", we know that

$$P(C) = .003 \quad P(+ \mid C) = .5 \quad P(+ \mid C') = .03$$

This suggests the following tree diagram:



Knowing that the sum of the probabilities from one point on the tree should add to 1, we can finish the tree diagram as follows:



The probability we are looking for is $P(C' \mid -)$. Notice that the tree diagram has $P(- \mid C')$, but not the reverse conditional probability that we are looking for. This is a sign we need to use Bayes' Rule. Let's find the appropriate form of Bayes' Rule. The definition of conditional probability applied to these events tells us

$$P(C' \mid -) = \frac{P(C' \cap -)}{P(-)}$$
$$P(- \mid C') = \frac{P(- \cap C')}{P(C')}$$

Solving each of these for the intersection yields

$$P(C' \mid -)P(-) = P(C' \cap -)$$
$$P(- \mid C')P(C') = P(- \cap C')$$

Since the intersection on the right-hand side is the same in each equation, we know the left-hand sides must be equal.

$$P(C' \mid -)P(-) = P(- \mid C')P(C')$$

Solving for $P(C' \mid -)$ gives

$$P(C' \mid -) = \frac{P(- \mid C')P(C')}{P(-)}$$

This is Bayes' Rule for this problem. Now we are ready to use the tree diagram. $P(- \mid C')$ and $P(C')$ are both labeled on the tree diagram. We can calculate $P(-)$ by following the branches on the tree diagram (multiply) that lead to a negative result, and then summing up the products from these branches.

$$P(-) = (.003)(.5) + (.997)(.97) = .96859$$

Putting these values into Bayes' Rule gives

$$P(C' \mid -) = \frac{(.97)(.997)}{.96859} \approx .9985$$

This means that is you test negative, the likelihood that you do not have colorectal cancer is 99.85%. The test is quite good at screening that you do not have the disease.

Practice

The probability of colorectal cancer can be given as .3%. If a person has colorectal cancer, the probability that the hemoccult test is positive is 50%. If a person does not have colorectal cancer, the probability that he still tests positive is 3%. What is the probability that a person who tests positive does have colorectal cancer?