8.1 Permutations

Question 1: How do you count choices using the Multiplication Principle?

Question 2: What is factorial notation?

Question 3: What is a permutation?

Question 1: How do you count choices using the Multiplication Principle?

Key Terms

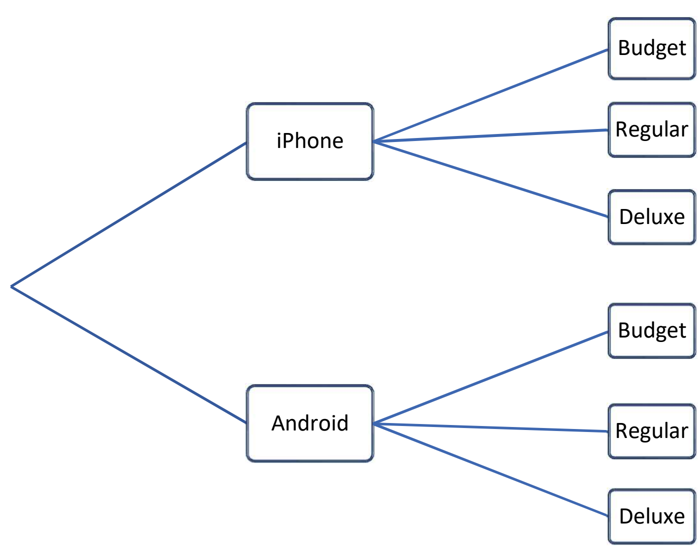
Multiplication Principle

Summary

Businesses are often faced with decisions in which they may make many choices. To count the number of ways these decisions are made, we can apply the **Multiplication Principle**. Let's see where this principle comes from through an example.

A small cellular provider gives its customers two choices of phones to use. They may use an iPhone or a phone that uses the Android operating system. In addition, the company offers three different calling plans: Budget plan, Regular plan, and the Deluxe plan. How many different choices of phone and calling plan does a customer have?

To answer this question, we can use a decision tree and list out the choices a customer may make in a decision tree.



A decision tree shows the different choices a customer makes when choosing a phone and plan. If we move left to right through the tree, we can list out each of the possibilities. The first set of branches lists the choice of phones and the second set of branches lists the plans:

iPhone with Budget plan	iPhone with Regular plan	iPhone with Deluxe plan
Android with Budget plan	Android with Regular plan	Android with Deluxe plan

By listing out each of the possibilities, we see that there are six possible phone/plan choices. The decision tree helps us to list out these possibilities. However, if we only need to know how many choices, we can multiply the number of choices for phones and plans.

$$\underline{\quad 2 \quad} \cdot \underline{\quad 3 \quad} = 6$$

Number of phones to choose from

Number of plans to choose from

This strategy is useful for determining the total number of choices even when there are a larger number of choices.

Multiplication Principle

Suppose we wish to know the number of ways to make *n* choices where there are

d_1 ways to make choice 1

d_2 ways to make choice 2

$\vdots$

d_n ways to make choice n

Then the total number of ways to make all of the choices is

$$d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

If we can enumerate the number of ways to make a series of decisions, the product of these numbers tells us how many ways there are to make this series of decision.

Notes

Guided Example 1	Practice
An online custom bicycle seller wishes to count the total number of different types of bicycles that are available through its website. The seller offers 4 different frame styles, 8 different fender colors, 10 different tire colors, 8 different wheel colors, 6 different pedal colors, and 12 different accessory colors. How many different bicycles can a customer order? **Solution** Each choice the customer must make leads to a different factor in the multiplication principle. $\underset{\text{frame style}}{\underline{4}} \cdot \underset{\text{fender color}}{\underline{8}} \cdot \underset{\text{tire color}}{\underline{10}} \cdot \underset{\text{wheel color}}{\underline{8}} \cdot \underset{\text{pedal color}}{\underline{6}} \cdot \underset{\text{accessory color}}{\underline{12}} = 184{,}320$ There are 184,320 different bicycles that can be ordered.	The owner of a Great Purchase, an online stereo store, wants to advertise that he has many different sound systems in stock. The store carries 6 different Blu Ray players, 10 different receivers, and 5 different speakers. Assuming a sound system consists of one of each, how many different sound systems can he advertise?

Guided Example 2	Practice
A company wants to have 3-digit phone extensions with the first digit not being zero. a. How many possible extensions are there? **Solution** Break the problem into three choices that need to be made. These choices indicate the number of ways to choose each of the three numbers in the extension. Since the first digit cannot be zero, there are 9 ways to choose the first number. The second and third numbers may contain zero, so there are 10 ways to choose each of those numbers. The total number of extensions is the product of these choices, $\underset{\text{first digit}}{\underline{9}} \cdot \underset{\text{second digit}}{\underline{10}} \cdot \underset{\text{third digit}}{\underline{10}} = 900$	A pin code consists of four digits from 0 to 9 at a bank. a. How many possible pin codes are there?
b. For moral and safety reasons, the company wants to exclude the extensions 911 and 666. How many possible 3-digit extensions are there without these exclusions? **Solution** There are 900 extensions that exclude 0 as the first digit. The extensions 911 and 666 are two specific extensions. Excluding those extensions leaves 898 extensions. c. How many possible 3-digit extensions are there if the first digit is not zero and digits may not be repeated? **Solution** When digits are not repeated, the number of choices for the second and third digits is reduced: $\underset{\text{first digit}}{\underline{9}} \cdot \underset{\text{second digit}}{\underline{9}} \cdot \underset{\text{third digit}}{\underline{8}} = 648$ For the second digit, we may now use a zero, but not the first digit. This yields 9 choices. For the third digit, we cannot use either of the first two digits giving 8 possible digits.	b. How any possible pin codes are there if the digits may not be repeated?

Question 2: What is factorial notation?

Key Terms

Factorial

Summary

When we apply the Multiplication Principle and do not allow repetition, the number of choices in each part of the product drops by 1. This leads to products like $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

This type of product occurs so often that it is assigned its own symbol.

Factorial Notation

For any positive integer *n*,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

The value of $0!$ is defined to be 1.

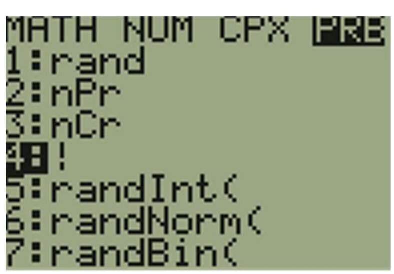
Let's look at how we might apply this to an application.

Suppose a production line requires six workers to carry out six different jobs. Each worker can only do one job at a time. Once a worker is selected for a job, the other jobs must be carried out by the remaining workers. To find the number of ways we can assign workers to jobs, calculate the product

$$\underset{\text{first job}}{\underline{6}} \cdot \underset{\text{second job}}{\underline{5}} \cdot \underset{\text{third job}}{\underline{4}} \cdot \underset{\text{fourth job}}{\underline{3}} \cdot \underset{\text{fifth job}}{\underline{2}} \cdot \underset{\text{sixth job}}{\underline{1}} = 720$$

The number of ways to make each choice drops by one in each factor since each worker can only do one job. In effect, we can't choose the same worker twice. This is often indicated by saying that we want to assign workers without repetition.

Instead of multiplying these factors out, we can utilize factorials and write it as $6!$. This may then be computed on a calculator such as a TI graphing calculator. The factorial symbol is located under the MATH button in the PRB submenu.



Notes

Guided Example 3

Evaluate each of the expressions involving factorials.

a. 8!

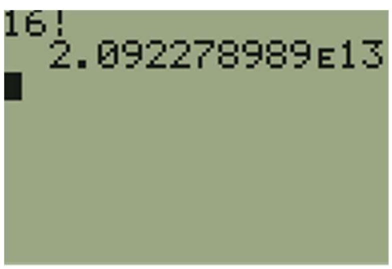
Solution Using the definition of factorial, this is equal to

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$$

b. 16!

Solution Using the definition of factorial, this is equal to

$$16 \cdot 15 \cdot 14 \cdots 3 \cdot 2 \cdot 1 \approx 2.092 \times 10^{13}$$



This is an incredibly huge number. The calculator uses scientific notation to display it. The E13 indicates $\times 10^{13}$. You would write this down by moving the decimal place 13 places to the right.

Practice

Evaluate each of the expressions involving factorials.

a. 5!

b. 20!

c. $\frac{100!}{98!}$ **Solution** Both of the numbers in the fraction are beyond most calculator's ability to calculate. However, if we use the definition of factorial we note an interesting pattern. $\frac{100!}{98!}=\frac{100\cdot 99\cdot 98\cdot 97\cdots 3\cdot 2\cdot 1}{98\cdot 97\cdots 3\cdot 2\cdot 1}$ Many of the factors on top match up with identical factors on the bottom. These may be reduced leaving $\frac{100!}{98!}=100\cdot 99=9900$	c. $\frac{82!}{80!}$

Question 3: What is a permutation?

Key Terms

Permutation

Summary

The term "**permutation**" refers to different arrangements of objects. We have already seen on example of a permutation in Question 2. When we allocated six workers among six jobs on a production line, we were counting the number of ways that we could arrange the workers among the jobs. This was a permutation of 6 workers taken 6 at a time. For permutations, we assume that the objects are arranged without repetition. In the context of the production line, this means that once a worker is given a job that worker cannot be assigned to another job.

How many ways can we assign six workers to four jobs without repetition? As before, we need to choose workers for each job.

$$\underset{\text{first job}}{\underline{6}} \cdot \underset{\text{second job}}{\underline{5}} \cdot \underset{\text{third job}}{\underline{4}} \cdot \underset{\text{fourth job}}{\underline{3}} = 360$$

This is the same pattern that resulted in factorial notation, but we are missing the last two factors. However, we can still write this product in terms of factorial notation.

$$6 \cdot 5 \cdot 4 \cdot 3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{6!}{2!} = \frac{6!}{(6-4)!}$$

The number in the numerator indicates the number of objects we are selecting from. The number in the denominator is the difference between the number of objects we are selecting from and the number we are selecting. This relationship leads to a general rule for permutations.

Permutations

An arrangement of *n* objects taken *r* at a time without repetition is called a permutation. The number of these arrangements is symbolized $P(n,r)$ and found with

$$P(n,r) = \frac{n!}{(n-r)!}$$

where $r \le n$. The symbol $P(n,r)$ is read "the permutation of *n* objects taken *r* at a time".

It can be confusing to use the letter *P* to indicate permutations and probability. For this reason, some textbooks will write $P_{n,r}$ or P_r^n instead of $P(n,r)$. In practice, this is less of a concern since the values in parentheses are numbers for permutations and events represented by capital letters in probability.

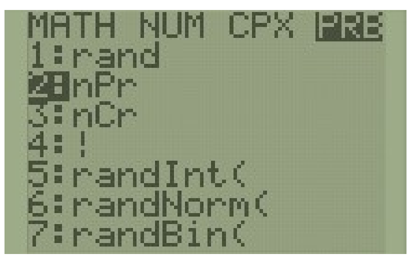
Permutation assume different arrangements of objects are counted separately. For instance, we could list the permutations of the letters CAT taken two at a time:

CA	AT	CT
AC	TA	TC

For permutations, order makes a difference, so CA is counted separately from AC. The number of permutations of three letters taken two at a time is $P(3,2)$ and is calculated as

$$P(3,2) = \frac{3!}{(3-2)!} = 6$$

Permutations may be calculated on a TI graphing calculator by entering the number of objects, nPr (press MATH and goto PRB submenu), and how many are taken at a time.



<u>Notes</u>

Guided Example 4

Evaluate each of the expressions below.

a. $P(10,5)$

Solution

$$P(10,5)=\frac{10!}{(10-5)!}=30240$$

b. $P(8,1)$

Solution

$$P(8,1)=\frac{8!}{(8-1)!}=8$$

Practice

Evaluate each of the expressions below.

a. $P(9,4)$

b. $P(6,1)$

Guided Example 5	Practice
The US Postal Service has used 5-digit zip codes since 1963 to help it to deliver mail in the US. a. How many zip codes are possible if there are no restrictions on the digits used? **Solution** Each digit can be any integer from 0 to 9. Since digits may be repeated, we need to use the Multiplication Principle to find the number of ways the numbers may be arranged: $\underset{\text{first digit}}{\underline{10}} \cdot \underset{\text{second digit}}{\underline{10}} \cdot \underset{\text{third digit}}{\underline{10}} \cdot \underset{\text{fourth digit}}{\underline{10}} \cdot \underset{\text{fifth digit}}{\underline{10}} = 10,000$ b. How many zip codes are possible if digits may not be repeated? **Solution** When digits are not repeated, we may use permutation to count the number of ways to select 5 digits from 10 digits: $P(10,5) = \frac{10!}{(10-5)!} = 30240$ Note that a permutation with P requires that we cannot repeat a selection.	In 1983, The US Postal Service introduced ZIP + 4 so that codes now consisted of 9 digits. a. How many zip codes are possible if there are no restrictions on the digits used? b. How many zip codes are possible if digits may not be repeated?

Guided Example 6

A city council consists of seven members. Two of those members are chosen to be mayor and vice mayor. In how many ways can this be done?

Solution We are interested in calculating the number of ways to arrange two council members taken two at a time. Since a council member cannot be mayor and vice mayor, repetition is not allowed. The number of arrangements is

$$P(7,2)=\frac{7!}{(7-2)!}=42$$

Practice

A business wants to select three employees to honor. One employee will be named employee of the year, another will be named volunteer of the year, and the last will be named advocate of the year. If there are 20 employees in the company and employees can only be selected for one honor, how many ways are there to honor employees?