The demand equation for telephones at one store is

$$p = D(q) = 200e^{-0.1q}$$

where p is the price (in dollars) and q is the quantity of telephones sold per week. Find the values of q and p that maximize revenue.

To be able to maximize the revenue function, we must first find the revenue function from the given function. Once we have it, we can apply the first derivative test to find the values that maximize revenue.

Find the Revenue Function

Revenue is related to demand by the simple equation,

$$\text{Revenue} = \text{Price} \times \text{Quantity}$$

In this problem, the demand function take in quantities q of phones sold per week and outputs the price p. Since the independent variable is q, it makes sense to do the same thing for the revenue function and think of it as

$$R(q) = D(q) \times q$$

Now we can substitute the function in the problem statement into this expression to give

$$\begin{aligned} R(q) &= 200e^{-0.1q} \times q \\ &= 200qe^{-0.1q} \end{aligned}$$

Notice that this is a function of the quantity q.

Maximize the Revenue Function

Now that we have the revenue function $R(q)$, we need to find the maximum on that revenue function. We'll do this by following this strategy:

1. Take the derivative of the revenue function.
2. Find where the derivative function is equal to 0 or undefined to locate the critical numbers.
3. Use the first derivative test to decide if the critical numbers are maximums, minimums or neither.
4. Find the values of q and p.

Step 1: The hardest step here is getting the derivative of the revenue function. Since the revenue is the product of $200q$ and $e^{-0.1q}$, we'll use the product rule to find the derivative.

$$u = 200q \quad \Rightarrow \quad u' = 200$$
$$v = e^{-0.1q} \quad \Rightarrow \quad v' = e^{-0.1q} \cdot (-0.1)$$

Notice that the derivative of v requires the chain rule. This gives the derivative

$$R'(q) = \underbrace{e^{-0.1q} \cdot 200 + 200q \cdot e^{-0.1q} \cdot (-0.1)}_{vu' \quad + \quad uv'}$$

Step 2: This sum is always defined, so we need to find where this is equal to zero to find the critical numbers. A little simplifying will help:

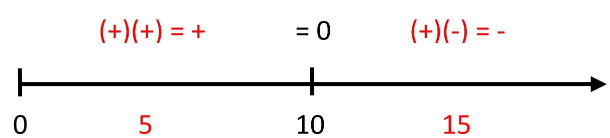
$$\begin{aligned} R'(q) &= e^{-0.1q} \cdot 200 + 200q \cdot e^{-0.1q} \cdot (-0.1) \\ &= 200e^{-0.1q}(1 - 0.1q) \end{aligned}$$

In this step I have factored the common $200e^{-0.1q}$ from both terms. For this product to equal 0, either $200e^{-0.1q}$ must equal zero or $1 - 0.1q$ must equal zero. This first factor is a decreasing exponential whose graph never crosses the x-axis so it is never equal to zero. The other factor is a linear factor and is equal to when

$$\begin{aligned} 1 - 0.1q &= 0 \\ -0.1q &= -1 \\ q &= \frac{-1}{-0.1} = 10 \end{aligned}$$

So there is only one critical number for this function $q = 10$.

Step 3: Now we need to determine whether this critical number is a maximum or a minimum. We'll do this by making a number line and applying the first derivative test. On this number line I'll only show values of q that are greater than or equal to 0 since q represents numbers of telephones sold per week.



When testing the numbers, keep in mind that $200e^{-0.1q}$ is always positive since its graph is always above the x-axis.

Since the function is increasing on the left side of $q = 10$ and decreasing on the right side of $q = 10$, the critical point is a relative maximum.

Step 4: We have the quantity $q = 10$ at the relative maximum. If we needed to know how much revenue there was at $q = 10$, we would find

$$R(10) = 200(10)e^{-0.1(10)} \approx \$735.76$$

However, this is not what we were asked to find. To find p when $q = 10$, we need to use the demand function $D(q)$:

$$p = D(10) = 200e^{-0.1(10)} \approx \$73.576$$

Although it makes more sense to round the price to two decimal places, I have written it with three decimal places so that we can notice that 10 telephones sold per week at $73.576 per phone yields revenue of $(\$73.576 \text{ per telephone})(10 \text{ telephones}) = \$735.76$. Notice that this is consistent with $R(10)$ from above.