Question 3: What is an average cost function?

When a business produces goods or services, it incurs costs associated with the production of these goods and services. Some of these costs are variable and change as productions changes. Items like raw materials, labor and the cost of utilities all vary as production changes and are called variables costs. Other costs, such as lease payments and insurance payments, are fixed. As production changes, these fixed costs do not vary.

The total cost is the sum of the variable and fixed cost,

$$\text{Total Cost} = \text{Variable Cost} + \text{Fixed Cost}$$

A simple model for the total cost is a linear model. In a linear model, the term containing the variable corresponds to the variable cost and the constant term corresponds to the fixed cost,

$$TC(Q) = \underbrace{aQ}_{\substack{\text{Variable} \\ \text{Cost}}} + \underbrace{b}_{\substack{\text{Fixed} \\ \text{Cost}}}$$

where Q units are produced. The constant a is the cost per unit and b is the fixed costs.

Another model for cost is a quadratic model,

$$TC(Q) = \underbrace{aQ^2 + bQ}_{\text{Variable Cost}} + \underbrace{c}_{\substack{\text{Fixed} \\ \text{Cost}}}$$

In this model, the terms with variables model the variable cost and the constant c models the fixed cost.

Businesses often seek to succeed by lowering their costs. However, this does not mean lowering the total cost function. The lowest total cost would be incurred by producing no units at all. Generally businesses seek to lower their average cost.

For a total cost function $TC(Q)$, where Q is the number of units produced, the average cost is defined

$$\overline{TC}(Q) = \frac{TC(Q)}{Q}$$

In other words, the average cost of producing Q units is the total cost of producing Q units divided by the number of units produced Q.

Let's look at a quick example. Suppose a company produces a single product called a solar toaster. It costs the company $20 to produce each solar toaster and the company has $5050 in daily fixed costs. If the company produces Q solar toasters each day, we can write out the total daily cost as

$$TC(Q) = 20Q + 5050 \text{ dollars}$$

Notice that this function has been defined on a daily basis, since the fixed cost is given on a daily basis. We could also define the total annual costs by modifying the variable description and the fixed costs appropriately. As long as the cost per unit is constant on an annual basis, this is also reasonable.

If we want to know the total daily cost of producing 100 toasters per day, we would compute

$$TC(100) = 20(100) + 5050 = 7050 \text{ dollars}$$

The average daily cost of producing 100 toasters per day is

$$\begin{aligned}\overline{TC}(100) &= \frac{7050 \text{ dollars}}{100 \text{ toasters}} \\ &= 70.50 \text{ dollars per toaster}\end{aligned}$$

This tells us that each solar toaster costs \$70.50 even though the cost of materials and labor is only \$20 per toaster. This is because the average cost takes into account the high daily fixed cost. However, the average cost to produce 200 toasters is

$$\begin{aligned}\overline{TC}(200) &= \frac{9050 \text{ dollars}}{200 \text{ toaster}} \\ &= 45.25 \text{ dollars per toaster}\end{aligned}$$

Even though the total daily cost to produce toasters is higher when 200 toasters are made each day, the average cost is lower.

We can use the average cost function

$$\overline{TC}(Q) = \frac{20Q + 5050}{Q}$$

to find the average cost at any production level.

> The derivative of the average cost function
>
> $$\overline{TC}'(Q) = \frac{d}{dQ}\left[\frac{TC(Q)}{Q}\right]$$
>
> is called the marginal average cost function.

This function is used to determine the rate at which the average cost function changes.

Example 5 Find the Average Cost Function

The cost of goods and services at Verizon are given by the function

$$TC(Q) = 490.268Q + 2367.072 \text{ million dollars}$$

where Q is the number of residential and wireless subscribers in millions.

a. Find the average cost function $\overline{TC}(Q)$.

Solution The average cost function is formed by dividing the cost by the quantity. in the context of this application, the average cost function is

$$\overline{TC}(Q) = \frac{TC(Q)}{Q}$$

Place the expression for the cost in the numerator to yield

$$\overline{TC}(Q) = \frac{490.268Q + 2367.072}{Q}$$

b. Find and interpret $\overline{TC}(50)$.

Solution The function value is obtained by substituting $Q = 50$ into the average cost function $\overline{TC}(Q) = \frac{490.268Q + 2367.072}{Q}$,

$$\begin{aligned} \overline{TC}(50) &= \frac{490.268(50) + 2367.072}{50} \\ &\approx 537.61 \end{aligned}$$

The numerator on the average cost function has units of millions of dollars and the denominator has units of millions of subscribers. Dividing the units yields

$$\frac{\text{units of } TC(Q)}{\text{units of } Q} = \frac{\cancel{\text{millions}} \text{ of dollars}}{\cancel{\text{millions}} \text{ of subscribers}} = \frac{\text{dollars}}{\text{subscriber}}$$

So $\overline{TC}(50) \approx 537.61$ means that when Verizon has 50 million subscribers, their average cost per subscriber is 537.61 dollars per subscriber.

c. Find the derivative of the average cost function $\overline{TC}'(Q)$.

Solution We can apply the Quotient Rule for Derivatives with $u = 490.268Q + 2367.072$ and $v = Q$. The derivatives of the numerator and denominator are

$$u = 490.268Q + 2367.072 \quad \rightarrow \quad u' = 490.268$$
$$v = Q \quad \rightarrow \quad v' = 1$$

Using the Quotient Rule, we get the derivative

$$\overline{TC}'(Q) = \frac{\overbrace{(Q)(490.268)}^{vu'} - \overbrace{(490.268Q + 2367.072)(1)}^{uv'}}{\underbrace{Q^2}_{v^2}}$$

$$= -\frac{2367.072}{Q^2}$$

d. Find and interpret the marginal average cost $\overline{TC}'(50)$.

Solution Set $Q = 50$ in $\overline{TC}'(Q) = -\dfrac{2367.072}{Q^2}$ to give

$$\overline{TC}'(50) = -\frac{2367.072}{50^2}$$
$$\approx -0.95$$

This rate indicates how fast the average cost is changing as the number of subscribers is increased. The units on this rate are

$$\frac{\text{units of } \overline{\text{TC}}(Q)}{\text{units of } Q} = \frac{\dfrac{\text{dollars}}{\text{subscriber}}}{\text{millions of subscribers}}$$

The value $\overline{TC}'(50) \approx -0.95$ tells us that at a subscriber level of 50 million, the average cost is decreasing by 0.95 dollars per subscriber for every 1 million additional subscribers.