#  Section 5.1 Percent and Change

**Getting Started** – How do you solve problems involving percent?

* How do you use percent to represent change?
* How do you solve applied problems with percent?

## **Getting Started** – How do you solve problems involving percent?

Key Terms

Percent

Summary

We have encountered percent in earlier chapters when working with exponential growth, exponential decay, and probability. In those sections, we needed to understand how to move between percents, decimals, and fractions. In this section, we will solve basic percent problems that involve sentences.

* What is 25% of 200?
* 5 is 40% of what number?
* 400 is what percent of 300?
* How much is 20% of 150?

To solve these problems, we will translate the sentence to an equation and solve the resulting equation. To translate the sentence, the dictionary below helps to match words with mathematics.

|  |  |
| --- | --- |
| Word | Math |
| what, how | unknown variable |
| is, are, equal, was, were | =  |
| of, times |  |

Notes

Guided Example 1 Practice

|  |  |
| --- | --- |
| What is 25% of 200?**Solution** To solve this percent problem, let’s break the sentence down and match the pieces with mathematical symbols.  The unknown may be represented by the variable *x*. This gives us the equation  The equation is already solved for the variable. Doing the multiplication gives us . We can check that in the original problem by replacing “what” with 50:50 is 25% of 200Using some number sense on this, we can realize that 100% of a number should be the number itself (200 is 100% of 200). This means that 25% of 200 should be smaller than 200. Since 50 is smaller than 200, it passes the number sense test. | What is 60% of 80? |

Guided Example 2 Practice

|  |  |
| --- | --- |
| 5 is 40% of what number?**Solution** Break the sentence down and match the pieces with mathematical symbols.  Write the unknown as *x* to get the equation  Solve this equation for *x* by dividing both sides of the equation by 0.40:  We can check that in the original problem by replacing “what number” with 12.5:5 is 40% of 12.5Since 40% is smaller than 100%, 5 must also be smaller than 12.5. Additionally, 50% of a number is the same as half of a number (50% of 12.5 is 6.25). So, 40% of a number must be a little smaller than 50% of a number. 5 is slightly smaller than 6.25 so the solution passes the number sense test. | 12 is 120% of what number? |

Guided Example 3 Practice

|  |  |
| --- | --- |
| 400 is what percent of 300?**Solution** To solve this percent problem, let’s break the sentence down and match the pieces with mathematical symbols.  The unknown may be represented by the variable x. This gives us the equation  Divide both sides by 300:  Since the question asked, “what percent”, we need to write  as a percent:   We can check that in the original problem by replacing “what percent” with 133%:400 is 133% of 300400 is bigger than 300 so we would expect the percent to be larger than 100%. So, the answer is reasonable. | 10 is what percent of 25? |

Guided Example 4 Practice

|  |  |
| --- | --- |
| How much is 20% of 150?**Solution** To solve this percent problem, let’s break the sentence down and match the pieces with mathematical symbols.  The unknown may be represented by the variable x. This gives us the equation  The equation is already solved for the variable. Doing the multiplication gives us . We can check that in the original problem by replacing “what” with 30:30 is 20% of 150This means that 20% of 150 should be smaller than 150. Since 30 is smaller than 150, it passes the number sense test. | How much is 250% of 120? |

## How do you use percent to represent change?

Key Terms

Percent change

Summary

Percentages are often used to describe how a quantity changes. **Percent change** is defined as

 

 where the base amount is the original amount of the quantity and new amount is the amount the quantity has changed to. This definition gives a decimal that must be changed to a percentage after calculating.

If the new amount is greater than the base amount, the percentage change is positive. If the new amount is less than the original amount, the percentage is negative.

As an example, suppose the price of a share in a company on the New York Stock Exchange doubles from $50 to $100. This indicates that the base amount is 50 and the new amount is 100. The percent change is

 

This is equivalent to a percentage of 100%.

In December of 2018, many companies suffered severe drops in their share prices. At the beginning of the month, the price of a share in Apple was $184.82. At the end of the month, the share price was $157.74. This means the base amount was 184.82 and the new amount was 157.74. This gives a percent change of

 

This corresponds to a percentage of -14.65%. Because the share price decreased, the percentage is negative as expected.

Guided Example 5 Practice

|  |  |
| --- | --- |
| The average price of a new car in 2000 was $24,750. By 2017, the average price of a new car was $31,400. What was the percent increase in the price of a new car from 2000 to 2017?**Solution** To find the percent change, use the definition of percent change,  The price in 2000 is the base amount, $24,750, The price increased to $31,400 so this is the new amount. Put these values in definition to give  Since we are finding the percent change, we need to rewrite the decimal as a percent. Move the decimal two places to the right (or multiply by 100) to give approximately 26.87%. | In November, Janet paid $69.52 for electricity. The following month, her electricity bill increased to $71.50. By what percent did the bill change? |

Guided Example 6 Practice

|  |  |
| --- | --- |
| In 2000, the average life expectancy in Angola was 57.9 years. In 2015, the life expectancy in Angola was 51.7 years What was the percent change of life expectancy in Angola from 2000 to 2015?**Solution** To find the percent change, use the definition of percent change,  Life expectancy in 2000 is the base amount, 57.9, Life expectancy decreased to 51.7 so this is the new amount. Put these values in definition to give  Since we are finding the percent change, we need to rewrite the decimal as a percent. Move the decimal two places to the right (or multiply by 100) to give approximately -10.71%. The percentage is negative because life expectancy has decreased. | In 2000, the average life expectancy in Benin was 60.7 years. In 2015, the life expectancy in Benin was 59.2 years What was the percent change of life expectancy in Benin from 2000 to 2015? |

## How do you solve applied problems with percent?

Key Terms

Percent Percent change

Summary

In the previous questions, we have seen two ways in which percentages may be used. In the first question, we looked at two numbers and a corresponding percent. In the second question, you were given a quantity that changes and then asked to calculate the percent change. The key difference is the change. If a question involves how a quantity changes, it is most likely a percent change problem. In this case, you use the definition

 

When a problem indicates that you need a percent of a quantity you should try to phrase it as a sentence that you can translate to an equation. Once this is accomplished, you can solve for the quantity indicated in the question.

Notes

Guided Example 7

|  |
| --- |
| The Consumer Price Index (CPI) is used to measure prices relative to some base price. The Bureau of Labor Statistics maintains the index which measures prices relative to the years 1982 to 1984. The base CPI is 100 and corresponds to average prices in 1982 to 1984. In November 2018, the CPI for food was 254.379. This means that prices for food in November 2018 were 254.379% of what they were in 1982 to 1984.1. If the price of a pound of bananas was $0.25 in 1982 to 1984, what was the price of a pound of bananas in November 2018?

**Solution** The key to solving this problem is to notice the phrase “prices for food in November 2018 were 254.379% of what they were in 1982 to 1984”. If we apply this to bananas, we get “banana prices in November 2018 were 254.379% of bananas prices in 1982 to 1984”. Let’s translate this phrase into mathematics:  Replacing the unknown with *x*, we get  This is already solved for *x* so working out the right-side yields . The price of a pound of bananas is $0.64 in November 2018. Note that the price has been rounded to two decimal places or the nearest cent.1. If the price of a box of Cheerios in November 2018 was $3.99, what was the price in 1982 to 1984?

**Solution** Rewrite the phrase in the problem statement as  If we put *x* in place of the unknown, we get  Divide both sides of the equation by 2.54379 to isolate *x*:  This tells us that the prices of Cheerios in 1982 to 1984 was approximately $1.57. |

Practice

|  |
| --- |
| In November 2018, the CPI for new vehicles was 145.826. Use this information to answer the questions below.1. In 1984, the list price of a new Toyota Corolla was $7778. Using the CPI, what would be the price of a new Corolla in November 2018?
2. In November 2018, the list price on a base model Toyota Tacoma was $25, 400. Using the CPI, find the list price on a new Tacoma in 1984.
 |

Guided Example 8

|  |
| --- |
| The rate of inflation is measured by calculating the percent change in the CPI between any two times. In November 2017, the overall CPI was 246.669. By November 2018, the overall CPI had increased to 252.038. What was the overall rate of inflation from November 2017 to November 2018.**Solution** Recall that percent change is defined to be   Applying this definition to our problem gives  Now put in the appropriate CPI numbers to yield  This equates to a rate of inflation of 2.2% |

Practice

|  |
| --- |
| In November 2017, the CPI for food was 250.871. By November 2018, the CPI for food had increased to 254.379. What was the rate of inflation for food from November 2017 to November 2018? |

# Section 5.2 Interest

* What is simple interest?
* What is compound interest?
* How do you use the compound interest formula to solve for different unknowns, such as present value, time, and interest rate of a loan?

## What is simple interest?

Key Terms

Simple interest Present value Future Value

Annual percentage rate Add-on loan Close-ended credit agreement

Installment loan

Summary

Money is not free to borrow! We will refer to money in terms of **present value** **P,** which is an amount of money at the present time, and **future value F,** which is an amount of money in the future. Usually, if someone loans money to another person in present value and are promised to be paid back in future value, then the person who loaned the money would like the future value to be more than the present value. That is because the value of money declines over time due to inflation. Therefore, when a person loans money, they will charge interest. They hope that the interest will be enough to beat inflation and make the future value more than the present value.

**Simple interest** is interest that is only calculated on the initial amount of the loan. This means you are paying the same amount of interest every year. An example of simple interest is when someone purchases a U.S. Treasury Bond.

The amount of interest paid is based on the interest rate or annual percent rate (APR) and the amount of time the money is borrowed. This relationship is described below.

**Simple Interest Formula**: 

where,

*F* = Future value

*P* = Present value

*r* = Annual percentage rate (APR) changed to a decimal

*t* = Number of years

Notice that the future value consists of two parts: the present value *P* and the interest *Prt.* Because of this relationship, you may also write the simple interest formula as

 

where the simple interest *I* is given by the formula .

In an add-on loan, interest is calculated using simple interest. In this type of loan, the borrower repays the future value of the loan amount spread out over payments. If the number of payments is fixed, the loan is called a **close-ended credit agreement** or **installment loan**. The amount of the payment *R* is

 

where *P* is the present value of the loan, *r* is the annual percentage rate, *t* is the length of the loan and *n* is the number of payments. We can also write this formula as

 

where the simple interest is .

Notes

Guided Example 1 Practice

|  |  |
| --- | --- |
| Chad got a student loan for $10,000 at an 8% annual simple interest rate. 1. How much does he owe after one year?

**Solution** Since Chad is borrowing $10,000, the present value is *P* = $10,000. The rate is 8% which corresponds to *r* = 0.08. We want the future value in one year, *t* = 1.After one year, he will owe $10,800.1. How much interest will he pay for that one year?

**Solution** Chad owes $10,800 after one year. This amount consists of the original amount of the loan and the interest on the loan.Subtract the original amount of the loan to find the interest paid on the loan. He will pay $10800 - $10000 = $800 in interest. | Lisa bought a computer for $1200 at 12% annual simple interest rate.1. How much does he owe after two years?
2. How much interest will he pay for two years?
 |

Guided Example 2 Practice

|  |  |
| --- | --- |
| Ben wants to buy a used car. He has $3000 but wants $3500 to spend. He invests $3000 into an account earning 6% annual simple interest. How long will he need to leave his money in the account to accumulate the $3500 he wants?**Solution** Ben presently has $3000 and wants it to grow to $3500 in the future. So, *P* = 3000 and *F* = 3500. Using the rate r = 0.06, we can calculate the find the time it takes to grow from the simple interest formula:Divide both sides by 0.06Subtract 1 from both sidesDivide both sides by 3000Carry out the calculation to find *t* on a TI 83/84 calculator.The time it takes to grow $3000 to $3500 is  years*Note: As shown above, wait to round your answer until the very last step so you get the most accurate answer.* | Sierra wants to go on a vacation that costs $4200. She invests $3800 into an account earning 8% annual simple interest. How long will he need to leave his money in the account to accumulate the $4200 he wants? |

Guided Example 3 Practice

|  |  |
| --- | --- |
| Judy invested $20,000 six years ago. She now has $26,000. What simple interest rate was her investment earning? **Solution** Judy has $20,000 initially which means the present value is *P* = 20,000. She ends up with $26,000 in six years so *t* = 6 and *F* = 26,000. To find the rate, solve for *r* in the simple interest formula  :  Divide both sides by 6Subtract 1 from both sidesDivide both sides by 20000On a TI graphing calculator, this may be computed as follows:An investment of $20,000 grows to $26,000 in six years at an interest rate of 5%. | Sergio doubled an investment of $10,000 in nine years.What simple interest rate was his investment earning?  |

Guided Example 4 Practice

|  |  |
| --- | --- |
| Jenni wants to buy a car that costs $23,000. She is prepared to pay $4,500 up front and finance the rest. The Ford Financing Company is offering her a 5-year simple interest loan at a rate of 2%. 1. How much will Jenni have paid at the end of 5 years?

**Solution** Jenni gives the down payment of $4500, so she now owes $23000 – 4500 = $18,500 on her car. The present value for her loan is *P* = 18500, the rate is *r* = 0.02, and the time is t = 5. We use the formula simple interest formula :  So, the future value of her loan is $20,350. But she paid $4,500 at the beginning, so at the end of 5 years, she paid a total of $20350 + 4500 **=** $24,850. 1. How much will Jenni have paid in interest at the end of 5 years?

**Solution** Jenni paid interest on the $18,500 she borrowed. To find out how much, we subtract the amount she borrowed from the future value of the loan. The interest she paid was  1. What is Jenni’s monthly payment?

**Solution** We do not count the down payment in the monthly payment, because the down payment was already paid! Take the future value of the loan and divide by the duration of the loan in months,  Hence Jenni will pay **$339.17 per month** on her car loan. Notice that when we are calculating quantities involving money, we round to the nearest cent. This is because we rarely in real life consider a tenth of a cent (three decimal places) or a hundredth of a cent (four decimal places). 1. What would Jenni’s monthly payment be if she made a down payment of $4,000?

**Solution** If Jenni makes a smaller down payment of $4000, then she needs to borrow $23000 – 4000 = $19000. The payment on the loan in this case is   Jenni’s new monthly payment is $348.33.  | Maria wants to buy a car that costs $40,000. She is prepared to pay $10,000 up front and finance the rest. The Desert Financial Credit Union is offering her a 5-year simple interest loan at a rate of 3.64%.  1. How much will Maria have paid at the end of 5 years?
2. How much will Maria have paid in interest at the end of 5 years?
3. What is Maria’s monthly payment?
4. What would Maria’s monthly payment be if she made a down payment of $4,000?
 |

## What is compound interest?

Key Terms

Compound interest

Summary

Most banks, loans, credit cards, etc. charge you compound interest, not simple interest. **Compound interest** is interest paid both on the original principal and on all interest that has been added to the original principal. Interest on a mortgage or auto loan is compounded monthly. Interest on a savings account can be compounded quarterly (four times a year). Interest on a credit card can be compounded weekly or daily!

|  |  |
| --- | --- |
| **Compounding type** | **Number of compounding periods per year** |
| Annually | 1 |
| Semiannually | 2 |
| Quarterly | 4 |
| Monthly | 12 |
| Daily | 365 |

When computing **compound interest**, interest is paid on the present value AND the interest accrued. Suppose you invest $1000 into an account that pays you 4% interest per year compounded semiannually for two years. Using compound interest, after the interest is calculated at the end of each period, then that amount is added to the total amount of the investment. Then the following period, the interest is calculated using the new total of the investment. When the interest is compounded semiannually, you only earn half the interest during each period.

|  |  |  |
| --- | --- | --- |
| **Year** | **Computation** | **Future Value of Investment** |
| 0 |  | $1000 |
| 0.5 |  | $1020 |
| 1 |  | $1040.40 |
| 1.5 |  | $1061.21 |
| 2 |  | $1082.43 |

Each row in the table is the previous row multiplied by 1.02. You can compute the future value of the investment for any number of periods using a table like the one above, but it can get tiresome if you are computing for many periods. In this case, it is easier to use the compound interest formula below.

**Compound Interest Formula**:  where,

*F* = Future value

*P* = Present value

*i* = interest rate per period

*n* = Number of periods

The interest rate per period is calculated by dividing the Annual percentage rate (APR) changed into a decimal by the number of compounding periods per year.

With compound interest, you earn interest on the present value as well as the interest during each period. In simple interest, you earn interest on the present value only in each period. Let’s compare a savings plan that pays 6% simple interest versus another plan that pays 6% annual interest compounded quarterly. If we deposit $8,000 into each savings account, how much money will we have in each account after three years?

|  |  |
| --- | --- |
| **6% Simple Interest***P* = $8,000, *r* = 0.06, *t* = 3 | **6% Interest Compounded Quarterly***P* = $8,000,  , *n* = 12 |
|  |  |
|  |  |

Notes

Guided Example 5 Practice

|  |  |
| --- | --- |
| Sophia’s grandparents bought her a savings bond for $200 when she was born. The interest rate was 3.28% compounded semiannually, and the bond would mature in 30 years. How much will Sophia’s bond be worth when she turns 30?**Solution** Match each quantity in the problem to its variable:*P* = $200, *r* = 0.0328,  ,Since you are compounding semiannually for 30 years, the number of periods is  or *n* = 60. Put these values into the compound interest formula to giveThe bond will be worth approximately $530.77 after 30 years. | Alvin purchased a Series I US Savings Bond for $5000. The Bond earns 2.83% interest compounded quarterly and matures in 10 years.How much will Alvin’s bond be worth in 10 years? |

## How do you use the compound interest formula to solve for different unknowns, such as present value, time, and interest rate of a loan?

Key Terms

Root Logarithm

Summary

The compound interest formula,

 

contains four different quantities. In previous examples, we were given three quantities: the present value *P*, the annual interest rate, the number of compounding periods in a year, and the number of periods *n*. We can substitute these values into the formula to find the future value *F*.

In general, we need four of the quantities in the compound interest formula to solve for the fifth quantity. To do this, we might need a few algebra tools to solve for a quantity in the power or quantities in the base of .

For instance, suppose we are given the equation

 

To solve for *x* in the base of the right side, we need to remove the exponent of 3. This is accomplished using a third root,  . Utilize the root by taking the third root of both sides of the equation:

 

Since  , 

Different roots are useful for removing powers from expressions. This helps us to solve for the rate *r* or the number of compounding periods in a year *m* in the compound interest formula.

To remove a fourth power, we would use a fourth root  . To remove a fifth power, we would use a fifth root  and so on. You can find different roots under the MATH menu on a TI graphing calculator.

 

Depending on your model of calculator, you may need to enter the type of root first or enter the type of root after pressing the root button. Experiment with the button by computing  and  to determine the proper keystrokes for your TI graphing calculator. You can also use these examples to determine the proper keystrokes on other types of calculators such as one found on your phone or online.

You can also solve for quantities in the power using logarithms. The shorthand log is a substitute for writing out the word logarithm. For our purposes, a logarithm is a mathematical process you can carry out on your calculator using the button labeled . We can take the logarithm of any positive number to yield a number.

The power property of logarithms helps us to solve for variables in powers. This property says that a power inside a logarithm may be moved outside of a logarithm:

 

You can check this by using the log button on your TI calculator.

In general,

 **Exponent Property of Logarithms**: 

Although you may not be familiar with logarithms, you can utilize this property to solve for exponents in equations. For instance, you can solve for the variable in the equation

 

by taking the logarithm of each side of the equation:

 

The Exponent Property of Logarithms allows you to move the exponent to a factor in front of the logarithm on the right side of the equation,

 

The value of each logarithm is simply a number, so we can isolate the variable by dividing both sides of the equation by :

 



Notice that you can also solve by converting to a logarithm. For this equation, the base is 1.02 and this base has an input of x and an output of 1.4. Converting to a logarithm reverses the relationship between input and output:

Input

Output

Input

Output

 

Base

Base

We can evaluate this logarithm using the logBASE command under the MATH button on a TI calculator.

 

When you solve an exponential equation, you have options. You can use the Exponent Property of Logarithms or convert to a logarithm directly.

Notes

Guided Example 6 Practice

|  |  |
| --- | --- |
| Suppose $5000 grows to $8300 in 7 years. What is the annual interest rate if interest is compounded semiannually?**Solution** Use the compound interest formula,   where the present value is , the future value is , and the number of periods is  or 14. Put these values into the compound interest formula and solve for *i*:  If the rate per period is 0.0369 and there are two periods per year (semiannual), then the nominal rate is  or approximately 7.37%. | Suppose $10,000 grows to $15,575 in 5 years. What is the annual interest rate if interest is compounded quarterly? |

Guided Example 7 Practice

|  |  |
| --- | --- |
| Find the present value​ if the future value is $14,520.35 and compounded annually at an annual rate of 1.256% for 6 years.**Solution** Use the compound interest formula,   where the future value is , the interest rate per period is  and the number of periods is . Put these values into the compound interest formula and solve for *P*:  To accumulate $14520.35, you would need to start with approximately $13472.63. | Find the present value​ if the future value is $26,500 and compounded quarterly at a annual rate of 3.75% for 20 years. |

Guided Example 8 Practice

|  |  |
| --- | --- |
| Find the time required for ​$5000 to grow to at least ​$9100 when deposited at 2​% compounded semiannually.**Solution** Since interest is being compounded semiannually, the future value is given by  The future value is , the present value is , and the rate per period is is . Put these values into the formula above to get  Divide both sides by 5000 to put the equation in exponential form:  Take the logarithm of each side and apply the Power Property of Logarithms,   To solve for *n*, divide both sides by  ,  In approximately 60.18 periods, the $5000 will have grown to $9100. Since interest is compounded semiannually, this is the same as  or 30.09 years.If we convert directly to a logarithm, we would write This gives the same solution as the Exponent Property of Logarithms. | Find the time required for ​$2000 to double when deposited at 8​% compounded monthly. This time is called the doubling time. |

# Section 5.3. Credit Cards

* How do you compute finance charges on a credit card using the unpaid balance method?
* How do you compute the average daily balance and find credit card finance charges?

## How do you compute finance charges on a credit card using the unpaid balance method?

Key Terms

Open-ended credit agreement Finance charge Unpaid balance

Summary

In Section 5.2 we calculated the interest on an installment loan by computing simple interest on the amount of the loan. The interest and amount borrowed is paid back over a fixed number of payments. In this question we will examine finance charges in an open-ended credit agreement.

In an open-ended credit agreement, the borrower is preapproved to borrow up to a certain amount and may borrow this amount repeatedly (and pay it off repeatedly). In credit card loans, you may borrow and pay off the loan plus interest each month. The interest on a credit card loan is called the finance charge.

The finance charge may be calculated on the unpaid balance at the end of the previous month. This method, called the unpaid balance method, uses simple interest to compute the finance charge. This means you will need to examine the charges and credits over the period of the credit card statement.

Several terms on the statement will need to be defined in order to find the unpaid balance.

**Previous balance** is the dollar amount you still owe the credit card company before the current billing cycle began.

When you make a purchase, the credit card company does not always record your purchase that day, it takes some time for the information to be processed. The **post date** is when your purchase (or payment) was applied to your balance.

The **Annual Interest Rate or Annual Percent Rate (APR)** is the simple interest rate you use to calculate the finance charge.

To compute the unpaid balance at the end of the billing period, we must start with the unpaid balance at the end of the previous period and

**Add** any finance charges

**Add** any purchases

**Subtract** any returns

**Subtract** any payments made

Notes

Guided Example 1 Practice

|  |  |
| --- | --- |
| Suppose that on January 1, you have a credit card balance of $620. On January 10, you purchase a magazine subscription for $20. On January 12, you return an item to Target for a credit of $34.50. On January 22, you make a payment of $500. On January 25, you make a purchase at Safeway of $112. The annual interest rate on your credit card is 21%. Calculate the finance charge for January that will appear on the next month’s statement using the unpaid balance method.**Solution** We will need to calculate two different finance charges: one for the outstanding balance of $620 for the previous month and another for the unpaid balance at the end of January.The finance charge for the previous month is calculated using the simple interest, *Prt*:December finance charge:  Since the interest rate is an annual rate, the time must also be in years. For a one-month period, the time is  year.To find the finance charge for January, we need to add the finance charge and purchases and subtract the returns and payments from the previous month’s unpaid balance:  Use simple interest to find January’s finance charge,January finance charge:  | Suppose that on June 1, you have a credit card balance of $840. On June 5, you make an Amazon purchase of $120. On June 15, you make another purchase of $52 at Walmart. On June 22, you make a payment of $600. On June 26, you make a return at Costco of $165. The annual interest rate on your credit card is 21%. Calculate the finance charge for June that will appear on the next month’s statement using the unpaid balance method. |

## How do you compute the average daily balance and find credit card finance charges?

Key Terms

Average daily balance

Summary

The average daily balance method calculates the finance charge using simple interest on the average daily balance. The average daily balance is calculated by adding the balances on each day of the billing period and dividing by the number of days in the billing period:

 

The number in the numerator is usually easier to calculate by utilizing multiplication. For instance, suppose the balance on January 1 through 9 is $620. Instead of writing $620 + $620 + $ 620 + $620 + $620 + $620 + $620 +$620 + $620 in the sum, we can write 9 ∙ $620 to simplify the calculation on top.

Once we have calculated the average daily balance, we can find the finance charge with *Prt* where *P* is the average daily balance. In other words,

 

Notes

Guided Example 2

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Suppose that on January 1, you have a credit card balance of $620. On January 10, you purchase a magazine subscription for $20. On January 12, you return an item to Target for a credit of $34.50. On January 22, you make a payment of $500. On January 25, you make a purchase at Safeway of $112. The annual interest rate on your credit card is 21%. Calculate the finance charge for January that will appear on the next month’s statement using the average daily balance method.**Solution** To find the finance charge using the average daily balance, you need to apply simple interest to the average daily balance. The average daily balance is the sum of the balances on each day of the boiling period divided by the number of days in the billing period. Let’s outline the balance on each day using a table.

|  |  |  |  |
| --- | --- | --- | --- |
| Day | Transaction | Change to Balance | Balance |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

Each line in the table indicates the days of the month with a particular balance, the transaction on the first day of the period, how the balance changes due to the transaction, and the resulting balance after the transaction.Let’s start with the balance on the first day of the billing period, January 1. The balance will remain $620 until the transaction on January 10. This information is placed in the first row of the table.

|  |  |  |  |
| --- | --- | --- | --- |
| Day | Transaction | Change to Balance | Balance |
| 1, 2, 3, 4, 5, 6, 7, 8, 9 | Initial Balance |  | **$620** |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

On January 10, a purchase is made that increases the balance by $20 resulting in a new balance of $620 + $20 or $640.

|  |  |  |  |
| --- | --- | --- | --- |
| Day | Transaction | Change to Balance | Balance |
| 1, 2, 3, 4, 5, 6, 7, 8, 9 | Initial Balance |  | **$620** |
| 10, 11 | Purchase subscription | +20 | **$640** |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

Continuing with the other transactions gives us the table below.

|  |  |  |  |
| --- | --- | --- | --- |
| Day | Transaction | Change to Balance | Balance |
| 1, 2, 3, 4, 5, 6, 7, 8, 9 | Initial Balance |  | **$620** |
| 10, 11 | Purchase subscription | +20 | **$640** |
| 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 | Return item to Target | -34.50 | **$605.50** |
| 22, 23, 24 | Payment | -500 | **$105.50** |
| 25, 26, 27, 28, 29, 30, 31 | Purchase at Safeway | +112 | **$217.50** |

Now we find the average daily balance by adding the balance on each day of the billing period and dividing by the number of days in the billing period:  where the balance has been rounded to the nearest penny. Notice that we have utilized multiplication in the calculation so that we do not need to add all 31 numbers directly.To find the finance charge, find the simple interest on the average daily balance of $475.94:  The time in years is found by dividing the number of days in the billing period by the number of days in a year. |

Practice

|  |
| --- |
| Suppose that on June 1, you have a credit card balance of $840. On June 5, you make an Amazon purchase of $120. On June 15, you make another purchase of $52 at Walmart. On June 22, you make a payment of $600. On June 26, you make a return at Costco of $165. The annual interest rate on your credit card is 21%. Calculate the finance charge for June that will appear on the next month’s statement using the average daily balance method. |

# Section 5.4 Annuities

* How do you calculate the future value of an ordinary annuity?
* What is a sinking fund?

## How do you calculate the future value of an ordinary annuity?

Key Terms

Annuity Simple ordinary annuity

Term

Summary

A sequence of payments or withdrawals made to or from an account at regular time intervals is called an **annuity**. The **term** of the annuity is length of time over which the payments or withdrawals are made. There are several different types of annuities. An annuity whose term is fixed is called an annuity certain. An annuity that begins at a definite date but extends indefinitely is called a perpetuity. If an annuities term is not fixed, it is called a contingent annuity. Annuities that are created to fund a purchase at a later date like some equipment or a college education are called sinking funds.

The payments for an annuity may be made at the beginning or end of the payment period. In an **ordinary annuity**, the payments are made at the end of the payment period. If the payment is made at the beginning of the payment period, it is called an **annuity due**. In this text we’ll only examine annuities in which the payment period coincides with the interest conversion period and the payments are made at the end of each period. This type of annuity is called a **simple ordinary annuity**.

Let’s look at an ordinary annuity that is certain and simple. By this, we mean an annuity over a fixed term whose payment period matches the interest conversion period. Additionally, the payments to the annuity are made at the end of the payment period. Suppose a payment of $1000 is made semiannually to the annuity over a term of three years. If the annuity earns 4% per year compounded semiannually, the payment made at the end of the first six-month period will accumulate

 

This means $1000 is multiplied by 1.02 five times, once for each of the remaining six-month periods.

The next payment also earns interest, but over 4 six-month periods. This payment has a future value of

 

This process continues until we have the future value for each payment.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|

|  |
| --- |
| First payment period |

|  |
| --- |
| Second payment period |

|  |
| --- |
| Third payment period |

|  |
| --- |
| Fourth payment period |

|  |
| --- |
| Fifth payment period |

|  |
| --- |
| Sixth payment period |

 | $1000 grows to  |
| $1000 grows to  |
| $1000 grows to  |
| $1000 grows to  |
| $1000 grows to  |
| $1000 |

The last payment occurs at the end of the last period and earns no interest. The sum of these amounts is

 

You can add the amount from each period to find the future value of the annuity, but this becomes tedious when there are many annuity payments. In this case, you can use the annuity formula to find the future value of the annuity.

**Future Value of an Ordinary Annuity**

If equal payments of *R* are made into an ordinary annuity for *n* periods at an interest rate of *i* per period, the future value of the annuity *F* is



For the ordinary annuity above, a payment of $1000 is made semiannually to the annuity over a term of three years. If the annuity earns 4% per year compounded semiannually, the future value is

 



Note that *i* is the interest rate per period. This means the annual interest rate must be divided by 2 since the payments are made semiannually (twice per year).

Notes

Guided Example 1 Practice

|  |  |
| --- | --- |
| An investor deposits $500 in a simple ordinary annuity at the end of each six-month payment period. This annuity earns 10% per year, compounded semiannually.1. Find the future value if payments are made for three years.

**Solution** Find the future value of this ordinary annuity using  In this case, , , and . This gives This is calculated in a TI Graphing Calculator as shown below. The six payments of $500 have earned  or $400.96 in interest over the life of the annuity.1. Find the future value if payments are made for 30 years.

**Solution** In this ordinary annuity, the term is much longer. Set , , and  in the formula for the future value of an annuity, we get This is calculated in a TI Graphing Calculator as shown below.1. How much interest is earned over the 30-year term in part b?

**Solution** Over the term of the annuity, sixty payments of $500 are made for a total of $30,000. This yields  or $146,791.86 in interest. | An employee deposits $100 in a simple ordinary annuity monthly. This annuity earns 8% per year, compounded monthly.1. Find the future value if payments are made for ten years.
2. Find the future value if payments are made for 35 years.
3. How much interest is earned over the 35-year term in part b?
 |

Notes

## What is a sinking fund?

Key Terms

Sinking fund

Summary

Annuities that are created to fund a purchase at a later date like some equipment or a college education are called sinking funds. In a sinking fund, the future value is known and another quantity in the annuity formula,

 

is being solved for. In the example below, a value for *F* is given and the payment *R* is calculated that leads to that future value.

Notes

Guided Example 2 Practice

|  |  |
| --- | --- |
| Suppose you want to accumulate $2,000,000 in a retirement account in 40 years. The retirement account averages an interest rate of 8% per year. How much would you need to deposit every two weeks (directly from your paycheck) to accumulate $2,000,000?**Solution** Since deposits are being made at the end of each two week period, this is an ordinary annuity where the future value is , the interest rate per period is , and the number of periods is  or 1040. Put the values into the ordinary annuity formula,  and work out the quantity in brackets on the right-hand side:Putting this value into the equation gives  Now divide each side by 7608.996665 to get the payment *R*,   Each payment would need to be approximately $262.85 to accumulate $2,000,000. | Suppose you want to have $25,000 in an account in 6 years to purchase a new vehicle. The account earns 3.25% per year. How much would you need to put into the account each month to accumulate $25,000? |

# Section 5.5 Amortization

* How do you calculate the present value of an annuity?
* How do you find the payment to pay off an amortized loan?
* What is an amortization schedule?

## How do you find the present value of an annuity?

Key Terms

Present value

Summary

We have been using the ordinary annuity formula,

 

to find the future value of payments made to an annuity. Often, we would like to know how much we would need to deposit all at once with compound interest to obtain the same future value.as making payments to an annuity. In this case, we want to know when the future value from compound interest,

 

is equal to the future of the annuity.

 

By setting the two right-hand sides equal to each other, we can determine the present value *P* of the annuity. This leaves us with the equation,

 

If we know the interest rate per period *i*, the payment *R*, and the number of periods *n*, we can solve for the present value *P* as illustrated in the examples below.

Guided Example 1

|  |
| --- |
| Find the present value of an ordinary annuity with payments of $10,000 paid semiannually for 15 years at 5% compounded semiannually.**Solution** We’ll use the formulato find the present value *P*. From the problem statement, we know thatPut these values into the formula and solve for PV: Isolate *P* using division.Work out the expression on each sideThis means that if we deposit $209,302.93 with compound interest or deposit $10,000 semiannually for 15 years, we will end up with the same future value of $439,027.03 (the number in blue from the annuity formula). |

Practice

|  |
| --- |
| Find the present value of an ordinary annuity with payments of $90,000 paid annually for 25 years at 8% compounded annually. |

## How do you find the payment to pay off an amortized loan?

Key Terms

Payment Amortization

Summary

Auto or home loans are often made to consumers so that they can afford a large purchase. In these types of loans, some amount of money is borrowed. Fixed payments are made to pay off the loan as well as any accrued interest. This process is called amortization.

In the language of finance, a loan is said to be amortized if the amount of the loan and interest are paid using fixed regular payments. From the perspective of the lender, the amount borrowed needs to be paid back with compound interest. From the perspective of the borrower, the amount borrowed, and interest is paid back via payments in an ordinary annuity:

Future Value with Compound Interest

Future Value of Ordinary Annuity

 

Suppose you want to borrow $10,000 for an automobile. Navy Federal Credit Union offers a loan at an annual rate of 1.79% amortized over 12 months. To find the payment, identify the key quantities in the formula:

 

Put these values into the payment formula to get

 

Now work out the expressions on the left and the expression in the brackets.



 

The payment has been rounded up to the nearest cent. This ensures that the loan will be paid off. This means that you pay slightly more than is needed. In practice, this is accounted for in an amortization schedule (also called an amortization table).

Notes

Guided Example 2 Practice

|  |  |
| --- | --- |
| Find the payment necessary to amortize a loan of $7400 at an interest rate of 6.2% compounded semiannually in 18 semiannual payments.**Solution** To find the payment, use the formulaIn this case,Put the values in the formula to give  Now work out the expression on each side and solve for *R* to give  This payment has been rounded up to the nearest cent. To find the total payments, multiply the amount of each payment by 18 to getTo find the total amount of interest paid, subtract the original loan amount from the total payments, | Find the payment necessary to amortize a loan of $25,000 at an interest rate of 8.4% compounded quarterly in 24 quarterly payments. |

## What is an amortization schedule?

Key Terms

Amortization schedule

Summary

An amortization schedule (also called an amortization table) records the portion of the payment that applies to the principal and the portion that applies to interest. Using this information, we can determine exactly how much is owed on the loan at the end of any period.

The amortization schedule generally has 5 columns and rows corresponding to the initial loan amount and the payments. The heading for each column are shown below.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |

To fill out the table, you need to carry out a sequence of steps to get each row of the table.

1. The first row of the table corresponds to the initial loan balance. Call this payment 0 and place the amount loaned in the column title “Outstanding Balance at the End of the Period”.
2. Go to the next line in the table and enter the payment calculated on the loan.
3. In the same row, use  to find the interest on the outstanding balance. Place this under the column titled “Interest in Payment”.
4. To find the “Amount in Payment Applied to Balance”, subtract the “Interest in the Payment” from the “Amount of Payment”.
5. To find the new “Outstanding Balance at the End of the Period”, subtract the “Amount in Payment Applied to Balance” from the “Outstanding Balance at the End of the Period” in the previous payment.

Fill out these quantities for all payments until the past payment. In the last payment, start by paying off the loan by making “Amount in Payment Applied to Balance” equal to the “Outstanding Balance at the End of the Period” in the second to last payment. This means the loan will be paid off resulting in the “Outstanding Balance at the End of the Period” for the final payment being 0. Finally, calculate the interest in the final payment and add it to the “Amount in Payment Applied to Balance” to give the final payment. Because of rounding in the payment, this may be slightly higher of lower than the other payments.

Let’s look at an example of a $10,000 for an automobile. Navy Federal Credit Union offers a loan at an annual rate of 1.79% amortized over 12 months. The amortization schedule below shows the calculation of the quantities for payment 1 and the last payment. Other payments follow a similar process.

**5.**

10000 – 826.52

Need to pay off the loan in the last payment

**3.**

$$ I=Prt=10000∙\frac{.0179}{12}∙1$$

Rounded to nearest cent

**1.** Starting balance

**2.** Calculated Payment

**4.**

841.44 – 14.92

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in the Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 10000 |
| 1 | 841.44 | 14.92 | 826.52 | 9173.48 |
| 2 | 841.44 | 13.68 | 827.76 | 8345.72 |
| 3 | 841.44 | 12.45 | 828.99 | 7516.73 |
| 4 | 841.44 | 11.21 | 830.23 | 6686.50 |
| 5 | 841.44 | 9.97 | 831.47 | 5855.03 |
| 6 | 841.44 | 8.73 | 832.71 | 5022.32 |
| 7 | 841.44 | 7.49 | 833.95 | 4188.37 |
| 8 | 841.44 | 6.25 | 835.19 | 3353.18 |
| 9 | 841.44 | 5.00 | 836.44 | 2516.74 |
| 10 | 841.44 | 3.75 | 837.69 | 1679.05 |
| 11 | 841.44 | 2.50 | 838.94 | 840.11 |
| 12 | 841.36 | 1.25 | 840.11 | 0.00 |

**4.**

841.48 – 14.92

1.25 + 840.11

Note that the interest has been rounded to the nearest cent. Different lenders may round the interest in different ways. Make sure you understand the rounding for the interest and the payment in order to obtain the corresponding amortization schedule.

Guided Example 3

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Suppose a loan of $2500 is made to an individual at 6% interest compounded quarterly. The loan is repaid in 6 quarterly payments.1. Find the payment necessary to amortize the loan.

**Solution** To find the payment on the loan, use the formula  For this problem, the interest rate per period is . The present value is  and the number of periods is . Using these values gives   Depending on how the rounding is done, this gives a payment of $438.81 or 438.82. For a calculated payment, the payment is often rounded to the nearest penny. However, many finance companies will round up to insure the last payment is no more than the other payments.1. Find the total payments and the total amount of interest paid based on the calculated monthly payments.

**Solution** The total payments (assuming the payment is rounded to the nearest penny) isThe total amount of interest is1. Find the total payments and the total amount of interest paid based on an amortization table.

**Solution** Making the amortization table takes several steps. Let me take it in pieces using the payment from above.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |

   The next row is filled out in a similar manner.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |

   Continue this process until the last row

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 |  |  |  |  |

After the fifth payment, we have $432.34 of principal left to pay in the final payment. So, this is the principal in the sixth payment. The interest is found by paying interest on the outstanding balance,  This gives a final payment of Now put these numbers into the amortization table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 | 438.83 | 6.49 | 432.34 | 0 |

Since the payments had been rounded to the nearest penny (rounded down), the final payment is slightly higher than the previous payments. Adding all the payments we get a total of $2632.88. Adding the interest amounts gives total interest of $132.88. |

Practice

|  |
| --- |
| Suppose a loan of $5000 is made to an individual at 4% interest compounded semiannually. The loan is repaid in 6 semiannual payments.1. Find the payment necessary to amortize the loan. Round the payment to the nearest penny.

 1. Find the total payments and the total amount of interest paid based on the calculated monthly payments.
2. Find the total payments and the total amount of interest paid based on an amortization table.
 |

**Chapter 5 Solutions**

Section 5.1

1. 48
2. 10
3. 40%
4. 300
5. About 2.8%
6. About -2.5%
7. $11342.35
8. 1.40%

Section 5.2

1. a. $1488, b. $288
2. Approximately 1.32 years
3. Approximately 11.1%
4. a. $45460, b. $5460, c. $591, d. $709.20
5. $6628.92
6. Rate per period is about 2.24% and the annual rate is about 8.96%
7. $12561.52
8. About 104.32 periods or 8.69 years

Section 5.3

1. $4.58
2. Average daily balance is $764.23 and the corresponding interest is $13.19

Section 5.4

1. a. $18294.60, b. $229388.25, c. Payments total to $42000 and interest totals to $187388.25
2. $314.94

Section 5.5

1. $960729.60 to nearest cent
2. $1336.80 to nearest cent
3. a. 892.63 to nearest cent, b. Total payments are 5355.78 with total interest 355.78 d. All numbers to nearest penny

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Payment Number** | **Amount of Payment** | **Interest in Payment** | **Amount in Payment Applied to Balance** | **Outstanding Balance at the End of the Period** |
| 0 |  |  |  | 5000 |
| 1 | 892.63 | 100.00 | 792.63 | 4207.37 |
| 2 | 892.63 | 84.15 | 808.48 | 3398.89 |
| 3 | 892.63 | 67.98 | 824.65 | 2574.24 |
| 4 | 892.63 | 51.48 | 841.15 | 1733.09 |
| 5 | 892.63 | 34.66 | 857.97 | 875.12 |
| 6 | 892.62 | 17.50 | 875.12 | 0 |

Total payments are 5355.77.

Total interest is 355.77.