

## Section 4.4 The Simplex Method and the Standard Minimization Problem

Question 1 – What is a standard minimization problem?

Question 2 – How is the standard minimization problem related to the dual standard maximization problem?

Question 3 - How do you apply the Simplex Method to a standard minimization problem?

Question 4 - How do you apply the Simplex Method to a minimization application?

Question 1 – What is a standard minimization problem?

### Key Terms

Standard maximization problem

### Summary

A standard minimization problem is a type of linear programming problem in which the objective function is to be minimized and has the form

$$w = d_1y_1 + d_2y_2 + \cdots + d_ny_n$$

where  $d_1, \dots, d_n$  are real numbers and  $y_1, \dots, y_n$  are decision variables. The decision variables must represent non-negative values. The other constraints for the standard minimization problem have the form

$$e_1y_1 + e_2y_2 + \cdots + e_ny_n \geq f$$

where  $e_1, \dots, e_n$  and  $f$  are real numbers and  $f \geq 0$ .

The standard minimization problem is written with the decision variables  $y_1, \dots, y_n$ , but any letters could be used as long as the standard minimization problem and the corresponding dual maximization problem do not share the same variable names.

### Notes

Guided ExamplePractice

Rewrite the linear programming problem so that it is a standard minimization problem.

$$\text{Minimize } w = 16y_1 + 14y_2 + 12y_3$$

subject to

$$y_1 + y_2 + y_3 \geq 6000$$

$$y_3 \leq \frac{2}{3}y_2$$

$$y_1 \geq 0.25(y_1 + y_2 + y_3)$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

**Solution** The objective function has the proper form. The inequality fits the form needed for a standard minimization. In this format the variables appear linearly on the left side of the inequality and are greater than or equal to a nonnegative number. Each of the inequalities must have this form to apply the Simplex Method.

The second constraint has variables on both sides of the inequality. To put it into the proper form, subtract  $y_3$  from both sides to give

$$0 \leq \frac{2}{3}y_2 - y_3$$

Flip flopping the sides and inequality give an equivalent inequality that is in the proper format:

$$\frac{2}{3}y_2 - y_3 \geq 0$$

To put the third inequality into the proper format, subtract  $0.25(y_1 + y_2 + y_3)$  from both sides to yield

$$y_1 - 0.25(y_1 + y_2 + y_3) \geq 0$$

Remove the parentheses and combine like terms to get

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Now add these modified constraints to the linear programming problem,

1. Rewrite the linear programming problem so that it is a standard minimization problem.

$$\text{Minimize } w = .06y_1 + .04y_2 + .02y_3$$

subject to

$$y_1 + y_2 + y_3 \geq 1000$$

$$y_3 \leq \frac{1}{2}y_2$$

$$y_2 \geq 0.5(y_1 + y_3)$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

<p>Minimize <math>w = 16y_1 + 14y_2 + 12y_3</math>  subject to</p> $y_1 + y_2 + y_3 \geq 6000$ $\frac{2}{3}y_2 - y_3 \geq 0$ $0.75y_1 - 0.25y_2 - 0.25y_3 \geq 0$ $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$ <p>Notice that all of the variables are on the left side of the greater than sign in the inequalities. Additionally, the right side of each inequality is nonnegative.</p>	
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Question 2 – How is the standard minimization problem related to the dual standard maximization problem?

### Key Terms

Dual problem          Transpose

### Summary

The linear programming problem

$$\text{Minimize } z = \frac{7}{4}y_1 + y_2$$

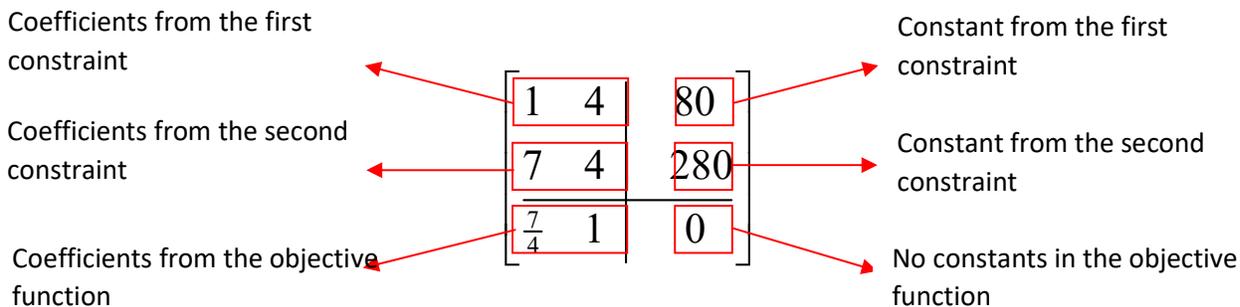
subject to

$$y_1 + 4y_2 \geq 80$$

$$7y_1 + 4y_2 \geq 280$$

$$y_1 \geq 0, y_2 \geq 0$$

is a standard minimization problem. The related dual maximization problem is found by forming a matrix before the objective function is modified or slack variables are added to the constraints. The entries in this matrix are formed from the coefficients and constants in the constraints and objective function:



To find the coefficients and constants in the dual problem, switch the rows and columns. In other words, make the rows in the matrix above become the columns in a new matrix,

$$\left[ \begin{array}{cc|c} 1 & 7 & \frac{7}{4} \\ 4 & 4 & 1 \\ \hline 80 & 280 & 0 \end{array} \right]$$

This new matrix is called the transpose of the original matrix. The values in the new matrix help us to form the constraints and objective function in a standard maximization problem:

1	7	$\frac{7}{4}$	→	$x_1 + 7x_2 \leq \frac{7}{4}$
4	4	1	→	$4x_1 + 4x_2 \leq 1$
80	280	0	→	Maximize $z = 80x_1 + 280x_2$

Notice the inequalities have switched directions since the dual problem is a standard maximization problem and the names of the variables are different from the original minimization problem. Putting these details together with non-negativity constraints, we get the standard maximization problem

$$\text{Maximize } z = 80x_1 + 280x_2$$

subject to

$$x_1 + 7x_2 \leq \frac{7}{4}$$

$$4x_1 + 4x_2 \leq 1$$

$$x_1 \geq 0, x_2 \geq 0$$

This strategy works in general to find the dual problem.

NotesGuided Example

Find the dual maximization problem to the standard minimization problem below.

$$\text{Minimize } w = 4y_1 + 6y_2 + 8y_3$$

subject to

$$5y_1 + 10y_2 + 12y_3 \geq 100$$

$$2y_1 + 3y_2 + 5y_3 \geq 300$$

$$-0.5y_1 + 0.75y_2 + 0.75y_3 \geq 0$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

**Solution** Form a matrix where the first three rows correspond to the three constraints and the fourth row corresponds to the objective function:

$$\left[ \begin{array}{ccc|c} 5 & 10 & 12 & 100 \\ 2 & 3 & 5 & 300 \\ -0.5 & 0.75 & 0.75 & 0 \\ \hline 4 & 6 & 8 & 0 \end{array} \right]$$

The transpose of this matrix is

Practice

1. Find the dual maximization problem to the standard minimization problem below.

$$\text{Minimize } w = 2y_1 + y_2$$

subject to

$$y_1 + y_2 \geq 1$$

$$2y_1 + 4y_2 \geq 3$$

$$y_1 \geq 0, y_2 \geq 0$$

$$\left[ \begin{array}{ccc|c} 5 & 2 & -0.5 & 4 \\ 10 & 3 & 0.75 & 6 \\ 12 & 5 & 0.75 & 8 \\ \hline 100 & 300 & 0 & 0 \end{array} \right]$$

Use this matrix to write out the dual standard maximization problem with the variables  $x_1, x_2, x_3$ , and  $z$ :

$$\text{Maximize } z = 100x_1 + 300x_2$$

subject to

$$5x_1 + 2x_2 - 0.5x_3 \leq 4$$

$$10x_1 + 3x_2 + 0.75x_3 \geq 6$$

$$12x_1 + 5x_2 + 0.75x_3 \geq 8$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Notes

Question 3 – How do you apply the Simplex Method to a standard minimization problem?

Key Terms

Summary

To solve a standard minimization problem with the dual maximum problem

1. Make sure the minimization problem is in standard form. If it is not in standard form, modify the problem to put it in standard form.
2. Find the dual standard maximization problem.
3. Apply the Simplex Method to solve the dual maximization problem.
4. Once the final simplex tableau has been calculated, the minimum value of the standard minimization problem's objective function is the same as the maximum value of the standard maximization problem's objective function.
5. The solution to the standard minimization problem is found in the bottom row of the final simplex tableau in the columns corresponding to the slack variables.

Notes

Guided Example

Use the Simplex Method to solve

$$\text{Minimize } w = 5y_1 + 2y_2$$

subject to

$$2y_1 + 3y_2 \geq 6$$

$$2y_1 + y_2 \geq 7$$

$$y_1 \geq 0, y_2 \geq 0$$

**Solution** Start by finding the dual maximization problem. The matrix for the minimization problem is

$$\left[ \begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 1 & 7 \\ \hline 5 & 2 & 0 \end{array} \right]$$

The transpose of this matrix is

$$\left[ \begin{array}{cc|c} 2 & 2 & 5 \\ 3 & 1 & 2 \\ \hline 6 & 7 & 0 \end{array} \right]$$

This gives a dual maximization problem

$$\text{Maximize } z = 6x_1 + 7x_2$$

subject to

$$2x_1 + 2x_2 \leq 5$$

$$3x_1 + x_2 \leq 2$$

$$x_1 \geq 0, x_2 \geq 0$$

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & z & & \\ \hline 2 & 2 & 1 & 0 & 0 & & 5 \\ 3 & 1 & 0 & 1 & 0 & & 2 \\ \hline -6 & -7 & 0 & 0 & 1 & & 0 \end{array}$$

The pivot is the entry in the second row, second column. Since it is already a 1, use the row operations  $-2R_2 + R_1 \rightarrow R_1$  and  $7R_2 + R_3 \rightarrow R_3$  to put zeros above and below the pivot:

$$\begin{array}{cccccc}
 x_1 & x_2 & s_1 & s_2 & z & \\
 \hline
 -4 & 0 & 1 & -2 & 0 & 1 \\
 3 & 1 & 0 & 1 & 0 & 2 \\
 \hline
 15 & 0 & 0 & 7 & 1 & 14
 \end{array}$$

Since there are no negative numbers in the indicator row, we can read the solution from this tableau. If any of the indicators in the bottom row had been negative, we would need to pick another pivot and carry out the steps to get ones and zeros in the pivot column. The solution from the final tableau for the maximization problem is  $x_1 = 0$ ,  $x_2 = 2$  and  $z = 14$ . The minimization problem's solution is found under the slack variable yielding  $y_1 = 0$ ,  $y_2 = 7$ , and  $w = 14$ .

### Practice

1. Use the Simplex Method to solve

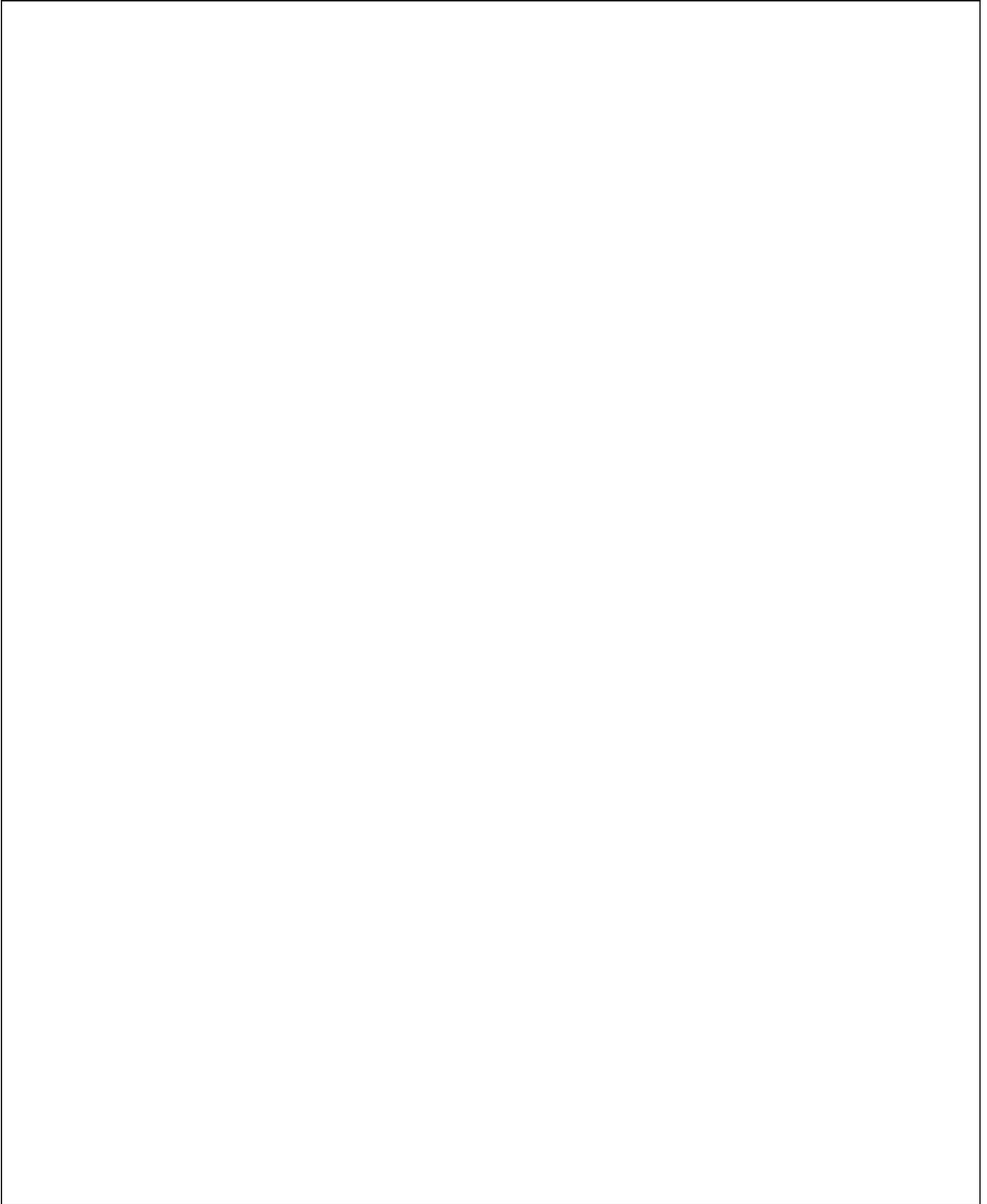
$$\text{Minimize } w = 8y_1 + 16y_2$$

subject to

$$y_1 + 5y_2 \geq 9$$

$$2y_1 + 2y_2 \geq 10$$

$$y_1 \geq 0, y_2 \geq 0$$



Question 4 – How do you apply the Simplex Method to a minimization application?

Key Terms

Summary

As with maximization applications, a minimization application can be broken down into two parts. First, you need to set up the application by writing out the variables, objective function, and constraints. Once the linear programming problem is written down and determined to be a standard minimization problem, we can solve the problem with the Simplex Method. For the minimization problem this will require you to find the dual maximization problem and then solve the maximization problem with the techniques from the previous questions. Remember to find the solution to the minimization problem in the indicator row and in the columns corresponding to the slack variables.

Notes

### Guided Example

The chemistry department at a local college decides to stock at least 800 small test tubes and 500 large test tubes. It wants to buy at least 2100 test tubes to take advantage of a special price. Since the small tubes are broken twice as often as the large, the department will order at least twice as many small tubes as large. If the small test tubes cost 15¢ each and large ones, made of a cheaper glass, cost 12¢ each, how many of each size should be ordered to minimize cost?

**Solution** Start by examining the question, “How many of each size should be ordered to minimize cost?”

This tells us two things:

1. We’ll be minimizing cost.
2. The variables are the number of test tubes of each type.

Start by defining exactly what the variables will represent:

$y_1$  : number of large test tubes to order

$y_2$  : number of small test tubes to order

In minimization problems we generally use  $y$ ’s as variables ( $x$ ’s for a maximization problem). Next, we need to relate the variables to the cost that we are minimizing. Locating the information about cost (small test tubes cost 15¢ each and large ones, made of a cheaper glass, cost 12¢ each), we can write

$$\text{Minimize } C = 12y_1 + 15y_2$$

where  $C$  is in cents.

With the objective function done, we can now move onto the constraints.

“Decides to stock at least 800 small test tubes and 500 large test tubes” means

$$y_2 \geq 800$$

$$y_1 \geq 500$$

“Buy at least 2100 test tubes to take advantage of a special price” means

$$y_1 + y_2 \geq 2100$$

Finally, the statement “the department will order at least twice as many small tubes as large” means

$$y_1 \geq 2y_2$$

This makes sense since it indicates that the small tubes are more than double the large tubes.

$$\text{Minimize } C = 12y_1 + 15y_2$$

subject to

$$y_2 \geq 800$$

$$y_1 \geq 500$$

$$y_1 + y_2 \geq 2100$$

$$y_1 \geq 2y_2$$

Before we can start using the simplex method, we need to rewrite the last constraint in the proper form,  $y_1 - 2y_2 \geq 0$ .

Now that have the linear programming problem in the standard linear form, we need to convert to the dual:

$$\begin{bmatrix} 0 & 1 & 800 \\ 1 & 0 & 500 \\ 1 & 1 & 2100 \\ 1 & -2 & 0 \\ 12 & 15 & 0 \end{bmatrix} \xrightarrow{\text{TRANSPOSE}} \begin{bmatrix} 0 & 1 & 1 & 1 & 12 \\ 1 & 0 & 1 & -2 & 15 \\ 800 & 500 & 2100 & 0 & 0 \end{bmatrix}$$

Converting this to a standard maximum problem yields

$$\text{Maximize } z = 800x_1 + 500x_2 + 2100x_3$$

subject to

$$x_2 + x_3 + x_4 \geq 12$$

$$x_1 + x_3 - 2x_4 \geq 15$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Adding slack variables to the constraints and rewriting the objective function we get

$$x_2 + x_3 + x_4 + s_1 = 12$$

$$x_1 + x_3 - 2x_4 + s_2 = 15$$

$$-800x_1 - 500x_2 - 2100x_3 + z = 0$$

The initial tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \hline 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & 0 & 1 & -2 & 0 & 1 & 0 & 15 \\ \hline -800 & -500 & -2100 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The pivot column is the third column since  $-2100$  is the most negative indicator. The pivot row is the first row since the ratio  $12/1$  is smallest. The pivot entry is already 1 so we need to make the rest of the column 0. To do this, we carry out  $-1R_1 + R_2 \rightarrow R_2$  and  $2100R_1 + R_3 \rightarrow R_3$ . The new matrix is

$$\left[ \begin{array}{cccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & -1 & 0 & -3 & -1 & 1 & 0 & 3 \\ \hline -800 & 1600 & 0 & 2100 & 2100 & 0 & 1 & 25200 \end{array} \right]$$

There is a negative in the indicator row, so pick it as the pivot column. Only the ratio  $3/1$  makes sense so the element in the second row, first column is the pivot. To make a zero below it,  $800R_2 + R_3 \rightarrow R_3$ .

The resulting matrix is

$$\left[ \begin{array}{cccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & -1 & 0 & -3 & -1 & 1 & 0 & 3 \\ \hline 0 & 800 & 0 & -300 & 1300 & 800 & 1 & 27600 \end{array} \right]$$

Now the fourth column and first row is the pivot (you can't choose a negative in the ratio denominator). To put zero in the rest of the column, carry out  $3R_1 + R_2 \rightarrow R_2$  and  $300R_1 + R_3 \rightarrow R_3$ .

The resulting matrix is

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array}$$

$$\left[ \begin{array}{cccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & 2 & 3 & 0 & 2 & 1 & 0 & 39 \\ \hline 0 & 1100 & 300 & 0 & 1600 & 800 & 1 & 31200 \end{array} \right]$$

The solution to the dual problem (the minimization problem) is found under the slack variables so  $y_1 = 1600$ ,  $y_2 = 800$  and  $z = 31200$ .

### Practice

1. An animal food must provide at least 54 units of vitamins and 60 calories per serving. One gram of soybean meal provides 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides 4.5 units of vitamins and 3 calories. One gram of grain provides 5 units of vitamins and 10 calories. A gram of soybean meal costs  $8\text{¢}$ , a gram of meat byproducts  $12\text{¢}$ , and a gram of grain  $10\text{¢}$ . What mixture of these three ingredients will provide the required vitamins and calories at minimum cost?

